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Economic Growth: Dynamic Interactions With International Trade And Global Environment

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Economic Growth: Dynamic Interactions with International Trade and Global Environment

by

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ABSTRACT

The thesis comprises four essays that, through theoretical analysis, explore the dynamic impact of capital accumulation on international trade (Chapters 1 and 2) and the dynamic interaction between economic activities and the global climate (Chapters 3 and 4).

Chapter 1 presents a dynamic Heckscher-Ohlin model in which agents' labor supply and investment decisions are endogenous. It is shown that analogues to the major results of the standard Heckscher-Ohlin model are valid in this model. Using the model developed in Chapter 1, Chapter 2 studies the determinant of comparative advantage in the long-run between countries with identical preferences. It is shown that a difference in the initial factor proportions causes trade to continue in the long-run. The country that is relatively capital abundant in the initial period remains relatively capital abundant and exports the relatively capital intensive good while the other country remains relatively labor abundant and exports the relatively labor intensive good in the long-run.

The goal of Chapters 3 and 4 is to offer an economic analysis on the greenhouse effect. Chapter 3 presents a two-sector dynamic model in which the productivity of the agriculture sector depends on weather temperature which in turn is affected by the production activities of the manufacture sector. Competitive equilibria in this model are analysed. Conditions under which the world economy and the global average temperature display a cyclical or chaotic time path are explored. The chapter concludes that the dynamic interaction between a stable natural system and a self-stabilizing market mechanism can, under certain conditions, lead to cyclical or even chaotic behavior. Using a two-country version of the model developed in Chapter 3, Chapter 4 shows that the optimal time path of outputs and temperature will converge to a unique steady state provided that consumers care enough about the future. The equilibrium outcome of a bargaining game where two countries negotiate an agreement on future consumption and production plans for the purpose of combating global warming is derived. It is shown that the agreement between the two countries can be implemented in decentralized economies by a system of taxes and transfers.

To Peng, my wife

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OVERVIEW

We live in an era in the history of mankind when there is growing economic and ecological interdependence among nations. The advances in communication and transportation technology have greatly enlarged and quickened the process of commodity trade across national frontiers. The importance of foreign trade to national economies has greatly increased for most nations in the post-war period. Economic growth and technological progress have also brought economic activities to an unprecedentedly large scale that is sufficient to affect the global ecosystem. The greenhouse effect is one of the well-known examples of the global environmental problems that are, as suggested by many scientists, caused by economic activities. International trade and global environment are two issues that are closely linked to economic growth.

This thesis comprises four essays that, through theoretical analysis, explore the dynamic impact of capital accumulation on international trade (Chapters 1 and 2) and the dynamic interaction between economic activities and the global climate (Chapters 3 and 4).

Capital accumulation is an important dimension of economic growth. The goal of Chapters 1 and 2 is to study how the standard international trade theory is affected by the introduction of endogenous capital accumulation into the Heckscher-Ohlin model. The theoretical model employed in the analysis is an open economy version of the standard two-sector neoclassical growth model with endogenous savings and endogenous labor supply. Chapter 1 is concerned with the issue that whether the introduction of endogenous factor supplies invalidates any of the basic results of the Heckscher-Ohlin model while Chapter 2 focuses on the long-run effects of capital accumulation on trade pattern.

The purpose of Chapters 3 and 4 is to offer an economic analysis on the greenhouse effect. Since the greenhouse effect is a form of externality, it is clear that a competitive equilibrium will not produce a globally optimal outcome. Thus the analysis is naturally divided into two segments: the competitive equilibrium and the global optimal solution. In Chapter 3 a one-country, two-sector general equilibrium model is constructed to study the characteristics of the time paths of the world economy and the global average temperature under the competitive equilibrium. In Chapter 4 the model is extended into a two-country setting to analyze the global optimal solution and a negotiation game whose outcome generates the global optimal solution.

In terms of methodology, the four chapters are unified by the approach of economic dynamics. While the conventional steady state analysis is conducted in the thesis, the focus of the thesis is on the dynamic time paths of the world economy.

Since this thesis covers a variety of topics, it is the author's expectation that most readers of this thesis will be interested in only a particular chapter of the thesis. Hence the thesis is edited in such a way that each chapter is self-contained and can be read without going through other chapters. The disadvantage of such an arrangement, however, is that those who decide to read this thesis in its entirety will have to endure the occasional repetitions between chapters.

CHAPTER 1

Endogenous Factor Supplies and the Heckscher-Ohlin Model

1.1. Introduction

The standard two-factor two-good model of international trade theory is traditionally analysed in a static framework. Since the sixties, some dynamic versions of this model have been developed to analyze problems related to the long-run equilibria in open economies (See, for example, Oniki and Uzawa, 1965; Bardhan, 1965; Stiglitz, 1970; Bertrand, 1975; Smith, 1977; Manning and Markusen, 1982). With the exceptions of Stiglitz (1970) and Manning and Markusen (1982), most of these dynamic models extend the standard model to a dynamic framework by simply assuming some fixed savings rule for the economy. The problem associated with this approach is that these *ad hoc* assumptions on savings behavior suppress or ignore the potentially important economic interactions between the saving behavior and the remaining endogenous variables in these models.

Another common feature shared by the dynamic Heckscher-Ohlin models is that the aggregate labor supply is assumed to be inelastic. This assumption suppresses the agents' problem of consumption-leisure choice. Furthermore, it blurs the concept of labor endowment. The labor endowment of an economy should refer to the size of labor force in the economy. But what is really meant by the labor endowment in these models is, in essence, the labor inputs used in the production. In reality, however, there is no reason to believe that the labor inputs will be a constant proportion of the size of the labor force irrespective of the economic environment. A more satisfactory and a more useful version of the Rybczynski Theorem and the Heckscher-Ohlin Theorem should, therefore, be stated in terms of the size of labor force rather than the labor inputs.

Trade theory literature that does incorporate endogenous labor supply has been based on static models (see, for example, Kemp and Jones, 1962; Martin, 1976). Furthermore, the labor supply functions in most of these models are assumed rather than being derived from a consumer's optimization problem.

In this paper, a standard neoclassical two-sector dynamic model with endogenous labor supply and endogenous investment decisions is presented. The goal of this paper is to study the validity of the basic results of the standard Heckscher-Ohlin model. In the literature, there are cases where the introduction of capital of some sort into the standard Heckscher-Ohlin model leads to a breakdown of its basic results (see, for example, Mainwaring, 1978; Steedman and Metcalfe, 1973; and Steedman and Metcalfe, 1977). Using a model with variable factor supplies, Martin (1978) suggests that the Heckscher-Ohlin Theorem could fail if the labor supply curve is backward-bending. He concludes that variable factor supplies provide another reason why the simple Heckscher-Ohlin model is not a very satisfactory predictor of comparative advantage.

In contrast to the results obtained by the aforementioned writers, in this chapter it is demonstrated that the Stolper-Samuelson Theorem and the Factor Price Equalization Theorem can be reproduced in this model with endogenous labor supply and endogenous savings. The Heckscher-Ohlin Theorem and the Rybczynski Theorem are restated and proven using the size of labor force rather than the labor inputs as the measure of labor endowment. The results in this chapter confirms Ethier's (1979) argument that the pessimistic conclusions about the Heckscher-Ohlin model are not due to the nature of capital but rather to other departures from the standard two-by-two neoclassical model. Furthermore, it is shown that the breakdown of the Heckscher-Ohlin Theorem in Martin's paper is due to the violation of the standard assumption that countries have identical preferences.

The short-run and long-run consequences of capital transfers and labor migration are studied in this chapter. It is shown that an expected exogenous increase in capital endowment will lead to an increase in the output of the capital intensive good and a decrease in the output of the labor intensive good before the endowment change takes place. As a result, the contemporaneous Rybczynski effects of an expected change in capital endowment are smaller in magnitude than that of an unexpected change. It is also shown that the Rybczynski effects of an exogenous change in capital endowment are reinforced by the endogenous changes in the labor supply in the short-run and in the long-run. The Rybczynski effects of a labor immigration, however, are dampened in the long-run by the higher investment level induced by the immigration.

This chapter is organized as follows. The basic physical environment of the economy is outlined in Section 1.2. A dynamic model in which a social planner maximizes social welfare over an infinite time horizon is presented in Section 1.3. Section 1.4 investigates the consequences of unexpected changes in the endowment of capital and in the size of labor force while Section 1.5 studies the effects of expected changes in capital endowment. The analogues of three basic results of the Heckscher-Ohlin model are proven in Section 1.6.

1.2. Economic Environment

In this section, the production and consumption side of the model are specified. Roughly speaking, the model studies an open economy with the standard two-by-two structure. There are two goods, a consumer good and a capital good. Both goods can be produced using two factors: capital and labor.

The formal specification of the model follows.

Time, denoted by t, is discrete and the horizon is infinite: $t \in \{0, 1, ...\}$.

Consider *an open economy* that is populated by a continuum of consumers. In each period, the economy is endowed with a fixed amount of time endowment, denoted by H. Time endowment is not tradeable. H measures the size of labor force in the economy. In period 0, the economy is endowed with capital stock K_0.

On the *production side* of the model, two goods are produced: a consumption good and a capital good, with quantities being denoted by y_1 and y_2. Both goods are tradeable and can be transported at zero cost. Consumption good is non-storeable. In each period, capital depreciates at a rate $\delta \in (0, \ 1]$.

The world commodity price of good i in period t is denoted by p_i $(i = 1, 2)$. Define $P_t \equiv \frac{p_{2t}}{p_{1t}}$. It is assumed that

(A1.1) $0 < P_t < \infty$ for all $t = 0, 1, 2, ...;$

which states that the relative commodity price is a positive finite number in all periods.

There is a fixed continuum of firms in each industry. Hence both industries are perfectly competitive. The two goods can be produced according to the following production functions: $y_{it} = F_i(k_{it}, l_{it})$ $(i = 1, 2)$, with y_{it} denoting the output of good i in period t, k_{it} the capital input into good i and l_{it} the labor devoted to the production of good i in period t. The nonnegative production functions satisfy:

(A1.2) $F_i(k_{it}, l_{it})$ is homogenous of degree one. $F_{ik} > 0$, $F_{il} > 0$, $F_i(0,0) = 0$, $F_{ik}(0, l_{it}) = +\infty$, $F_{il}(k_{it}, 0) = +\infty$. $F_{ikk} < 0$, $F_{ill} < 0$.

In this model, a good, say good i, is defined to be more capital intensive relative to good j if $\frac{k_i}{l_i} > \frac{k_j}{l_j}$ for any equilibrium wage-rental ratio. A standard assumption of the Heckscher-Ohlin model which will be maintained also in this model is that

(A1.3) there is no factor intensity reversal in production technology.

The total amount of capital stock in period t is denoted by K_t. Capital is perfectly mobile between sectors. The investment made in period t, denoted by I_t, will become productive in period $t + 1$. Hence the law of motion of capital is $K_{t+1} = (1 - \delta)K_t + I_t$. It is assumed that

(A1.4) there exists $\bar{K} > 0$ such that for any $k_{2t} < K_t$ and $l_{2t} < H$, $F_2(k_{2t}, l_{2t}) < \delta K_t$ if $K_t > \bar{K}$; and there exist some $k_{2t} < K_t$ and $l_{2t} < H$ such that $F_2(k_{2t}, l_{2t}) > \delta K_t$ if $K_t < \bar{K}$.

(A1.4) states that it is not possible for a self-sufficient economy to maintain capital stocks above $\bar{K}$.

Another standard assumption of the Heckscher-Ohlin model is that

(A1.5) the country produces both commodities in all periods[(1)].

On the *consumer side* of the economy, all consumers are identical. A representative consumer owns the country's intial capital stock K_0 and time endowment H. In each period he divides H into leisure and labor. His preference over consumption of the good and leisure is given by the utility function: $\sum_{t=0}^{\infty} \beta^t [U(C_t) + G(Z_t)]$, where $\beta \in (0,1)$ is the consumer's time discount factor, and C_t and Z_t denote his period t consumption of good 1 and leisure, respectively.

(A1.6) Both $U(\cdot)$ and $G(\cdot)$ are continuous, increasing, bounded below, and strictly concave functions with $U'(0) = +\infty$ and $G'(0) = +\infty$.

Firms and consumers are endowed with perfect foresight when making their decisions.

1.3. Social Planner's Problem

Consider a social planner who seeks to maximize the representative consumer's utility over an infinite time horizon by choosing appropriate levels of aggregate consumption, investment, labor efforts, leisure and outputs. The social planner is a price-taker in the worl' ommodity market. The social planner's problem can be written as

$$\max_{\{C_t, Z_t, I_t, k_{1t}, l_{1t}, k_{2t}, l_{2t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [U(C_t) + G(Z_t)]; \tag{1.3.1}$$

subject to

$$p_{1t}C_t + p_{2t}I_t = p_{1t}F_1(k_{1t}, l_{1t}) + p_{2t}F_2(k_{2t}, l_{2t}) \tag{1.3.2}$$

$$K_t = k_{1t} + k_{2t} \tag{1.3.3}$$

$$H = l_{1t} + l_{2t} + Z_t \tag{1.3.4}$$

$$K_{t+1} = I_t + (1-\delta)K_t \tag{1.3.5}$$

and $0 \leq K_0 \leq \bar{K}$.

Equation (1.3.2) distinguishes this open economy model from a standard closed economy model. Instead of demand equalling supply for both commodities, it requires that the value of consumption and investment equal the value of outputs of both goods in each period. (1.3.3) and (1.3.4) represent the full employment conditions of capital and labor in period t. In particular, (1.3.4) states that the total time available in each period is divided into working and leisure. (1.3.5) is the law of motion of capital stock.

Define

$$T(K_t, K_{t+1}; P_t) \equiv \max_{C_t, Z_t, k_{1t}, l_{1t}, k_{2t}, l_{2t}} \{U(C_t) + G(Z_t) \mid (1.3.2), (1.3.3), (1.3.4), (1.3.5)\}. \tag{1.3.6}$$

It can be verified that $T(K_t, K_{t+1}, P_t)$ is continuous and concave in (K_t, K_{t+1}) by assumption (A1.2) a: ⁴ (A1.6). Furthermore, the range of $T(K_t, K_{t+1}; P_t)$ is bounded due to assumptions (A1.1), (A1.4) and (A1.6).

Define

$$V(K_t;\{P_s\}_{s=t}^{\infty}) \equiv \max_{\{K_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} T(K_s, K_{s+1}; P_s), \quad t = 0, 1, 2, \cdots \tag{1.3.7}$$

By applying the result in Bertsekas (1976), we know that there is a unique bounded solution to the following Bellman's equation:

$$V(K_t;\{P_s\}_{s=t}^{\infty}) = \max_{K_{t+1}} T(K_t, K_{t+1}; P_t) + \beta V(K_{t+1};\{P_{s+1}\}_{s=t}^{\infty}). \tag{1.3.8}$$

Furthermore, the solution to (1.3.8), combined with the solution to (1.3.6), solves the social planner's problem (1.3.1)-(1.3.5).

The solution to the social planner's problem is characterized by the following four efficiency conditions:

$$\begin{aligned} U_c(C_t)P_t &= \beta V_k(K_{t+1};\{P_s\}_{s=t+1}^{\infty}) \\ &= \beta U_c(C_{t+1})[F_{1k}(K_{t+1} - k_{2t+1}; l_{1t+1}) + P_{t+1}(1-\delta)] \end{aligned} \tag{1.3.9}$$

$$U_c(C_t)F_{1l}(K_t - k_{2t}, l_{1t}) = G_z(H - l_{1t} - l_{2t}) \tag{1.3.10}$$

$$F_{1k}(K_t - k_{2t}, l_{1t}) = P_t F_{2k}(k_{2t}, l_{2t}) \tag{1.3.11}$$

$$F_{1l}(K_t - k_{2t}, l_{1t}) = P_t F_{2l}(k_{2t}, l_{2t}) \tag{1.3.12}$$

Since $C_t = F_1(K_t - k_{2t}; l_{1t}) + P_t F_2(k_{2t}; l_{2t}) + P_t(1-\delta)K_t - P_t K_{t+1}$, given K_t, (1.3.9)-(1.3.12) are four equations in four unknowns, K_{t+1}, l_{1t}, k_{2t}, l_{2t}. Equation (1.3.9) is a standard optimality condition governing investment. It states that the current utility loss due to an extra unit of investment must equal the discounted future utility obtained from this unit of investment. (1.3.10) illustrates that the marginal utility loss of working must equal the marginal benefit of working. (1.3.11) and (1.3.12) are the familiar efficiency conditions for capital market and labor market, respectively.

By Prescott and Mehra (1980), it can be established that the solution to the above social planner's problem can be supported as a competitive equilibrium in this economy of homogeneous households and firms.

1.4. Unexpected Change in Factor Endowments

The well-known Rybczynski Theorem reveals the relationship between production and an exogenous change in factor endowments. Sections 1.4 and 1.5 are devoted to the analysis of the Rybczynski effects of changes in factor endowments (H and K_t) in this dynamic model. Since H measures the size of labor force, a change in H can be interpreted as a migration of labor at the beginning of the planning horizon. Similarly, an exogenous increase in the period t capital stock, K_t, can be considered to be the result of a transfer of capital from the rest of the world to the home country.

The standard Rybczynski Theorem considers the effects of endowment changes in one period on outputs of the same period. In a dynamic model with perfect foresight, however, an expected endowment change in period t may affect the period $t-1$ investment level, which in turn affects period t choice variables. Therefore, to find the equivalence of the Rybczynski Theorem in this dynamic model, we should consider the effects of unexpected changes in factor endowments. In other words, we should consider changes in the initial endowments, H and K_0, on the period 0 choice variables so that the influences of the past history are suppressed. Hence in this section we study the case of unexpected changes in factor endowment. The effects of an expected change in capital endowment are analyzed in Section 1.5.

The proofs of many results in this chapter involve long mathematical expressions obtained from comparative statics. For the ease of exposition, only the signs of the derivatives are presented in the proofs. The detailed expressions of these derivatives are relegated to Appendix I.

Performing comparative statics on the efficiency condition (1.3.9)–(1.3.12) corresponding to period 0 yields:

$$\frac{\partial l_{10}}{\partial H} \begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.1}$$

$$\frac{\partial l_{20}}{\partial H}\begin{cases} >0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ <0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.2}$$

$$\frac{\partial k_{20}}{\partial H}\begin{cases} >0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ <0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.3}$$

(1.4.1)-(1.4.3) states that an increase in time endowment in period 0 leads to an increase in labor and capital inputs into the labor intensive good and a decrease in the two inputs into the capital intensive good in that period.

The reverse will happen if there is an increase in capital endowment in period 0, as revealed by:

$$\frac{\partial l_{10}}{\partial K_0}\begin{cases} >0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ <0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.4}$$

$$\frac{\partial k_{10}}{\partial K_0}\begin{cases} >0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ <0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.5}$$

$$\frac{\partial l_{20}}{\partial K_0}\begin{cases} >0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ <0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.6}$$

$$\frac{\partial k_{20}}{\partial K_0}\begin{cases} >0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ <0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.7}$$

Proposition 1.4.1. *(The Rybczynski Theorem). Assume (A1.1)-(A1.6). Given the commodity prices, an unexpected increase in a factor endowment (time or capital) in period 0 will lead to an increase in the output of the commodity using that factor intensively and a decrease in the output of the other commodity in the same period.*

Proof:

$$\frac{\partial F_i(k_{i0},l_{i0})}{\partial H} = F_{ik}\frac{\partial k_{i0}}{\partial H} + F_{il}\frac{\partial l_{i0}}{\partial H}\begin{cases} >0, & \text{if } \frac{k_{i0}}{l_{i0}} < \frac{k_{j0}}{l_{j0}}; \\ <0, & \text{if } \frac{k_{i0}}{l_{i0}} > \frac{k_{j0}}{l_{j0}}. \end{cases} \tag{1.4.8}$$

$$\frac{\partial F_i(k_{i0},l_{i0})}{\partial K_0} = F_{ik}\frac{\partial k_{i0}}{\partial K_0} + F_{il}\frac{\partial l_{i0}}{\partial K_0}\begin{cases} >0, & \text{if } \frac{k_{i0}}{l_{i0}} > \frac{k_{j0}}{l_{j0}}; \\ <0, & \text{if } \frac{k_{i0}}{l_{i0}} < \frac{k_{j0}}{l_{j0}}. \end{cases} \tag{1.4.9}$$

where $i,j = 1,2$, $i \neq j$.

Q.E.D.

Proposition 1.4.2. *Assume (A1.1)-(A1.6). Given the commodity prices, an unexpected increase in time endowment in period 0 will increase the exports (or decrease the imports) of the relatively labor intensive good and increase the imports (or decrease the exports) of the relatively capital intensive good in the same period.*

Proof:

$$\frac{\partial(F_1(0)-C_0)}{\partial H}\begin{cases} >0, & \text{if } \frac{k_{1u}}{l_{1u}} < \frac{k_{2u}}{l_{2u}}; \\ <0, & \text{if } \frac{k_{1u}}{l_{1u}} > \frac{k_{2u}}{l_{2u}}. \end{cases} \tag{1.4.10}$$

$$\frac{\partial(F_2(0)-I_0)}{\partial H}\begin{cases} >0, & \text{if } \frac{k_{1u}}{l_{1u}} > \frac{k_{2u}}{l_{2u}}; \\ <0, & \text{if } \frac{k_{1u}}{l_{1u}} < \frac{k_{2u}}{l_{2u}}. \end{cases} \tag{1.4.11}$$

Q.E.D.

Proposition 1.4.3. *Assume (A1.1)-(A1.6). Given the commodity prices, an unexpected increase in capital endowment in period 0 will lead to an increase in the exports (or a reduction in the imports) of the relatively capital intensive good and an increase in the imports (or a reduction in the exports) of the relatively labor intensive good in the same period if the capital good is relatively capital intensive or if the capital good is relatively labor intensive and $\delta = 1$. The effects of an unexpected increase in capital endowment on trade pattern is ambiguous if the capital good is relatively labor intensive and $\delta \in (0,\ 1)$.*

Proof:

$$\frac{\partial(F_1(0)-C_0)}{\partial K_0}\begin{cases} >0, & \text{if } \frac{k_{1u}}{l_{1u}} > \frac{k_{2u}}{l_{2u}} \text{ and } \delta = 1; \\ \gtreqless 0, & \text{if } \frac{k_{1u}}{l_{1u}} > \frac{k_{2u}}{l_{2u}} \text{ and } \delta \in (0,\ 1); \\ <0, & \text{if } \frac{k_{1u}}{l_{1u}} < \frac{k_{2u}}{l_{2u}}. \end{cases} \tag{1.4.12}$$

$$\frac{\partial(F_2(0)-I_0)}{\partial K_0}\begin{cases} >0, & \text{if } \frac{k_{1u}}{l_{1u}} < \frac{k_{2u}}{l_{2u}}; \\ <0, & \text{if } \frac{k_{1u}}{l_{1u}} > \frac{k_{2u}}{l_{2u}} \text{ and } \delta = 1; \\ \gtreqless 0, & \text{if } \frac{k_{1u}}{l_{1u}} > \frac{k_{2u}}{l_{2u}} \text{ and } \delta \in (0,\ 1). \end{cases} \tag{1.4.13}$$

Q.E.D.

Proposition 1.4.3 suggests that the introduction of a non-perishable capital good into the model creates ambiguities about the effects of an unexpected change in capital endowment on the trade pattern of the country. Intuitively this is because when $\delta < 1$, an increase in capital endowment raises not only the current capital stock but also the future capital stocks. As a result, the demand for investment is reduced. If the capital good is relatively labor intensive, output of the capital good is reduced as well. The net effects on the export of the capital good is, hence, ambiguous.

It is easy to establish that the period 0 investment is increased as a result of an increase in capital or time endowment, ie,

$$\frac{\partial K_1}{\partial H} > 0 \tag{1.4.14}$$

$$\frac{\partial K_1}{\partial K_0} > 0 \tag{1.4.15}$$

Moving (1.4.14) and (1.4.15) t ($t = 1, 2, \cdots$) periods ahead gives $\frac{\partial K_{t+1}}{\partial H} > 0$ and $\frac{\partial K_{t+1}}{\partial K_t} > 0$.

Proposition 1.4.4. *(The Long-run Rybczynski Theorem). Assume (A1.1)-(A1.6). Given the commodity prices, an unexpected increase in capital endowment in period 0 will increase the production of the relatively capital intensive good and reduce the production of the relatively labor intensive good in all subsequent periods. In the long-run, the Rybczynski effect of an increase in time endowment on outputs are dampened by higher investment level induced by the increase. As a result, the long-run effect of an increase in time endowment on outputs is ambiguous.*

Proof:

$$\frac{\partial F_i(t)}{\partial K_0} = \frac{\partial F_i(t)}{\partial K_t}\frac{\partial K_t}{\partial K_{t-1}}\frac{\partial K_{t-1}}{\partial K_{t-2}}\cdots\frac{\partial K_1}{\partial K_0}\begin{cases} > 0, & \text{if } \frac{k_{it}}{l_{it}} > \frac{k_{jt}}{l_{jt}}; \\ < 0, & \text{if } \frac{k_{it}}{l_{it}} < \frac{k_{jt}}{l_{jt}}, \end{cases} \tag{1.4.16}$$

where $i, j = 1, 2$, $i \neq j$.

Moving (1.4.8) t periods ahead gives:

$$\left.\frac{\partial F_i(k_{it}, l_{it})}{\partial H}\right|_{K_t}\begin{cases} > 0, & \text{if } \frac{k_{it}}{l_{it}} < \frac{k_{jt}}{l_{jt}}; \\ < 0, & \text{if } \frac{k_{it}}{l_{it}} > \frac{k_{jt}}{l_{jt}}. \end{cases} \tag{1.4.17}$$

which states that, given the capital stock, an augmentation in time endowment increases the output of the labor intensive good in all periods.

However, an increase in time endowment also raises investment level in each period, which, from the period t version of (1.4.9), has the opposite effect of reducing the output of the labor intensive good. The net effect is thus ambiguous:

$$\frac{\partial F_i(k_{it}, l_{it})}{\partial H} = \left.\frac{\partial F_i(k_{it}, l_{it})}{\partial H}\right|_{K_t} + \sum_{j=0}^{t-1} \frac{\partial F_i(k_{it}, l_{it})}{\partial K_t} \cdots \frac{\partial K_{t-j+1}}{\partial K_{t-j}} \frac{\partial K_{t-j}}{\partial H} \gtrless 0 \quad (1.4.18)$$

Q.E.D.

Proposition 1.4.5. *Assume (A1.1)-(A1.6). Given the commodity prices, an unexpected increase in capital endowment in period 0 will increase the exports (or decrease the imports) of the relatively capital intensive good and increase the imports (or decrease the exports) of the relatively labor intensive good in all subsequent periods if the capital good is relatively capital intensive or if $\delta = 1$ and the capital good is relatively labor intensive.*

Proof:

$$\frac{\partial(F_1(t) - C_t)}{\partial K_0} = \frac{\partial(F_1(t) - C_t)}{\partial K_t} \frac{\partial K_t}{\partial K_{t-1}} \frac{\partial K_{t-1}}{\partial K_{t-2}} \cdots \frac{\partial K_1}{\partial K_0}$$
$$\begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}} \ \& \ \delta = 1; \\ \gtrless 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}} \ \& \ \delta \in [0, 1); \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}. \end{cases} \quad (1.4.19)$$

$$\frac{\partial(F_2(t) - I_t)}{\partial K_0} = \frac{\partial(F_2(t) - I_t)}{\partial K_t} \frac{\partial K_t}{\partial K_{t-1}} \frac{\partial K_{t-1}}{\partial K_{t-2}} \cdots \frac{\partial K_1}{\partial K_0}$$
$$\begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}; \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}} \ \& \ \delta = 1; \\ \gtrless 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}} \ \& \ \delta \in [0, 1). \end{cases} \quad (1.4.20)$$

Q.E.D.

It is easy to verify that the long-run effects of an unexpected change in time endowment in period 0 on trade pattern is ambiguous. With the assistance of above results, the effects of an unexpected change in endowments on national income and consumption can be established.

Proposition 1.4.6. *Assume (A1.1)-(A1.6). Given the commodity prices, an unexpected increase (decrease) in initial capital or time endowment will lead to an increase (decrease) in national income, investment as well as the consumption of the good and leisure in all periods.*

Proof:

$$\frac{\partial(y_{10} + P_0 y_{20})}{\partial H} > 0 \tag{1.4.21}$$

$$\frac{\partial C_0}{\partial H} > 0 \tag{1.4.22}$$

$$\frac{\partial Z_0}{\partial H} > 0 \tag{1.4.23}$$

$$\frac{\partial(y_{10} + P_0 y_{20})}{\partial K_0} > 0 \tag{1.4.24}$$

$$\frac{dC_0}{dK_0} > 0 \tag{1.4.25}$$

$$\frac{\partial Z_0}{\partial K_0} > 0 \tag{1.4.26}$$

Furthermore, combining (1.4.14)-(1.4.15) and the period t version of (1.4.24)-(1.4.26) gives:

$$\frac{\partial(y_{1t} + P_t y_{2t})}{\partial x} > 0 \tag{1.4.27}$$

$$\frac{\partial C_t}{\partial x} > 0 \tag{1.4.28}$$

$$\frac{\partial Z_t}{\partial x} > 0 \tag{1.4.29}$$

where $x = H, K_0$.

Q.E.D.

Proposition 1.4.7. *Assume (A1.1)-(A1.6). Given the commodity prices, an unexpected increase (decrease) in time endowment will lead to an increase (decrease) in the contemporaneous aggregate labor supply but not necessarily proportionally. The changes in future labor supply are ambiguous. An increase (decrease) in initial* *capital endowment will unambiguously reduce (increase) the aggregate labor supply in all periods.*

Proof: Define the total labor supply $L_t \equiv l_{1t} + l_{2t}$.

$$\frac{\partial L_0}{\partial H} > 0. \tag{1.4.30}$$

But

$$\frac{\partial(\frac{L_0}{H})}{\partial H} \gtrless 0. \tag{1.4.31}$$

In the case of an unexpected change in capital endowment, however,

$$\frac{\partial L_0}{\partial K_0} < 0. \tag{1.4.32}$$

The long-run effects on labor supply are:

$$\frac{\partial L_t}{\partial H} = \frac{\partial L_t}{\partial H}\bigg|_{K_t} + \sum_{j=0}^{t-1} \frac{\partial L_t}{\partial K_t} \cdots \frac{\partial K_{t-j+1}}{\partial K_{t-j}} \frac{\partial K_{t-j}}{\partial H} \gtrless 0 \tag{1.4.33}$$

$$\frac{\partial L_t}{\partial K_0} = \frac{\partial L_t}{\partial K_t} \cdot \frac{\partial K_t}{\partial K_{t-1}} \frac{\partial K_{t-1}}{\partial K_{t-2}} \cdots \frac{\partial K_1}{\partial K_0} < 0. \tag{1.4.34}$$

Q.E.D.

In the standard Heckscher-Ohlin model, the total labor supply is assumed to be independent of changes in capital endowment. In this model with endogenous labor supply we see that labor supply will go down as a result of a capital inflow. Intuitively, this is because the increase in capital endowment raises the per capita national income which leads to the increase in the consumption of leisure.

One innovation of this new version of the Rybczynski Theorem is that the Rybczynski effect is larger when changes in the labor supply are taken into account. The traditional Rybczynski Theorem states that when there is an increase in capital endowment, the output of the capital intensive good will increase while the output of the labor intensive good will decrease, holding the labor supply constant. If the reduction in the labor supply is taken into consideration, there will be a further increase in the output of the capital intensive good and a further reduction in the output of the labor intensive good. Therefore, *the endogenous change in labor supply magnifies the traditional Rybczynski effects of a change in capital endowment.*

1.5. Expected Change in Capital Endowment

In this section, we study the case where a change in capital endowment is anticipated by agents in the economy. Suppose that the representative consumer in the country knows that in period t the country will receive a transfer of capital good in the amount of $\Delta K_t (> 0)$ as, say, international aid. If $\Delta K_t < 0$, then the country will transfer $-\Delta K_t$ amount of capital good to the rest of the world (eg, the home country is a donor of international aid.). In other words, if the representative consumer invests I_{t-1} ($\equiv K_t - (1-\delta)K_{t-1}$) in period $t-1$, the capital stock in period t will be $K_t + \Delta K_t$. Hence the transfer of capital in period t will affect the representative agent's decisions on not only the choice variables of period t and onward but also the period $t-1$ choice variables.

Performing comparative statics on the optimality conditions (1.3.9)-(1.3.12) corresponding to period $t-1$, one obtains,

$$\frac{\partial k_{2t-1}}{\partial \Delta K_t} = -\frac{\partial k_{1t-1}}{\partial \Delta K_t} \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \tag{1.5.1}$$

$$\frac{\partial l_{1t-1}}{\partial \Delta K_t} \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \tag{1.5.2}$$

$$\frac{\partial l_{2t-1}}{\partial \Delta K_t} \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \tag{1.5.3}$$

Proposition 1.5.1. *Assume (A1.1)-(A1.6). Given the commodity prices, an expected exogenous increase in capital stock in period $t(> 0)$ will lead to lower levels of investment, labor supply and national income but higher level of consumption of* *the good and leisure in period $t-1$. The reduction in investment in period $t-1$ does not fully offset the amount of capital that the country receives in period t. The aggregate capital stock in period t is thus higher than the level that would be without the transfer of capital.*

Proof:

$$\frac{\partial I_{t-1}}{\partial \Delta K_t} = \frac{\partial K_t}{\partial \Delta K_t} < 0. \tag{1.5.4}$$

But

$$\frac{\partial (K_t - \Delta K_t)}{\partial \Delta K_t} = \frac{\partial K_t}{\partial \Delta K_t} + 1 > 0. \tag{1.5.5}$$

Using the above results, one can easily obtain

$$\frac{\partial Z_{t-1}}{\partial \Delta K_t} = -\frac{\partial (l_{1t-1} + l_{2t-1})}{\partial \Delta K_t} > 0. \tag{1.5.6}$$

$$\frac{\partial (y_{1t-1} + P_{t-1} y_{2t-1})}{\partial \Delta K_t} = \frac{\partial F_1(t-1)}{\partial \Delta K_t} + P_{t-1} \frac{\partial F_2(t-1)}{\partial \Delta K_t} < 0 \tag{1.5.7}$$

$$\frac{\partial C_{t-1}}{\partial \Delta K_t} = \frac{\partial (y_{1t-1} + P_{t-1} y_{2t-1} - P_{t-1} I_{t-1})}{\partial \Delta K_t} > 0 \tag{1.5.8}$$

Q.E.D.

Proposition 1.5.2. *Assume (A1.1)-(A1.6). Given the commodity prices, an expected exogenous inflow of capital good will lead to an increase in the output of the capital intensive good and a decrease in the output of the labor intensive good before the flow takes place.*

Proof:

$$\frac{\partial F_1(t-1)}{\partial \Delta K_t} = F_{1k} \frac{\partial k_{1t}}{\partial \Delta K_t} + F_{1l} \frac{\partial l_{1t}}{\partial \Delta K_t} \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \tag{1.5.9}$$

$$\frac{\partial F_2(t-1)}{\partial \Delta K_t} = F_{2k} \frac{\partial k_{2t}}{\partial \Delta K_t} + F_{2l} \frac{\partial l_{2t}}{\partial \Delta K_t} \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \tag{1.5.10}$$

Q.E.D.

In the case of an unexpected inflow of capital good, the capital stock in that period is increased by the full amount of the capital that flows in. But this is not the case for an expected capital inflow in period $t(>0)$ since investment level in the previous period is lower. Therefore,

Proposition 1.5.3. *Assume (A1.1)-(A1.6). Given the commodity prices, the contemporaneous and long-run effects of an expected exogenous change in capital endowment in period $t(>0)$ is in the same direction as that of an unexpected change in capital endowment but with smaller magnitudes.*

Proof: Follows from (1.5.4) and (1.5.5).

Q.E.D.

1.6. The Analogues of Three Basic Theorems

Besides the Rybczynski Theorem, the remaining three basic results of the Heckscher-Ohlin model are the Factor Price Equalization Theorem, the Stolper-Samuelson Theorem and the Heckscher-Ohlin Theorem. In this section, it is shown that the first two results still hold in this model with endogenous labor supply. Furthermore, it is proven that the Heckscher-Ohlin Theorem can be generalized by measuring the relative factor abundance in terms of the capital/time ($\frac{K}{H}$) ratio rather than the capital/labor inputs ($\frac{K}{L}$) ratio. In addition, it is shown that a current price change has no effect on future returns to capital and labor.

Proposition 1.6.1. *(The Factor Price Equalization Theorem). Assume (A1.1)-(A1.6). With free trade, the equalization of the commodity prices in any period leads to the equalization of factor prices in that period.*

Proof: Rewrite (1.3.11) and (1.3.12):

$$P_t = \frac{F_{1k}(\frac{k_{1t}}{l_{1t}},1)}{F_{2k}(\frac{k_{2t}}{l_{2t}},1)} = \frac{F_{1l}(\frac{k_{1t}}{l_{1t}},1)}{F_{2l}(\frac{k_{2t}}{l_{2t}},1)} \tag{1.6.1}$$

These are the same conditions as those used by Samuelson (1949) in his original proof of this theorem.

Q.E.D.

Proposition 1.6.2. *(The Stolper-Samuelson Theorem). Assume (A1.1)-(A1.6). If the price of the labor intensive good rises relative to the price of the other good in period t, the real return to labor will unambiguously increase while the real return to capital will unambiguously decrease in that period irrespective of the endogenous changes in the capital stock and the labor supply induced by the price change. The converse is true if the relative price of the capital intensive good rises.*

Proof: Using the results of comparative statics, one can find that

$$\frac{\partial(\frac{k_{1t}}{l_{1t}})}{\partial P_t}\Big|_{K_t} = \frac{l_{1t}\frac{\partial k_{1t}}{\partial P_t} - k_{1t}\frac{\partial l_{1t}}{\partial P_t}}{l_{1t}^2} \begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}}; \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}. \end{cases} \tag{1.6.2}$$

$$\frac{\partial(\frac{k_{2t}}{l_{2t}})}{\partial P_t}\Big|_{K_t} = \frac{l_{2t}\frac{\partial k_{2t}}{\partial P_t} - k_{2t}\frac{\partial l_{2t}}{\partial P_t}}{l_{2t}^2} \begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}}; \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}. \end{cases} \tag{1.6.3}$$

(1.6.2)-(1.6.3) state that an increase in the period t price of the capital intensive good will, given the aggregate capital stock K_t, reduce the capital/labor ratio in the production of both goods while an increase in the price of the labor intensive good will raise this ratio.

Period t capital stock, however, may be changed as a result of the expected change in the relative price, P_t. Since

$$\frac{\partial(\frac{k_{it}}{l_{it}})}{\partial K_t} = 0, \qquad i = 1, 2, \tag{1.6.4}$$

then

$$\begin{aligned} \frac{d(\frac{k_{it}}{l_{it}})}{dP_t} &= \frac{\partial(\frac{k_{it}}{l_{it}})}{\partial P_t}\Big|_{K_t} + \frac{\partial(\frac{k_{it}}{l_{it}})}{\partial K_t}\frac{\partial K_t}{\partial P_t} \\ &= \frac{\partial(\frac{k_{it}}{l_{it}})}{\partial P_t}\Big|_{K_t} \end{aligned} \tag{1.6.5}$$

It is common knowledge that the marginal product of labor (capital) of a linearly homogeneous production function is a function of the capital/labor ratio $(\frac{k_{it}}{l_{it}})$. Furthermore, by the assumption of concavity,

$$\frac{dF_{ik}(\frac{k_{it}}{l_{it}},1)}{d(\frac{k_{it}}{l_{it}})} < 0 \tag{1.6.6}$$

$$\frac{dF_{il}(\frac{k_{it}}{l_{it}},1)}{d(\frac{k_{it}}{l_{it}})} > 0 \tag{1.6.7}$$

where $i = 1, 2$.

Define w_t as the wage of labor and r_t as the rental of capital in period t. It is obvious that in a competitive economy $\frac{w_t}{p_{it}} = F_{il}(t)$ and $\frac{r_t}{p_{it}} = F_{ik}(t)$ $(i = 1, 2)$. Therefore,

$$\frac{d(\frac{w_t}{p_{it}})}{dP_t} = \frac{dF_{il}(\cdot)}{dP_t} = \frac{dF_{il}(\cdot)}{d(\frac{k_{it}}{l_{it}})} \cdot \frac{d(\frac{k_{it}}{l_{it}})}{dP_t} \begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}}; \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}. \end{cases} \tag{1.6.8}$$

$$\frac{d(\frac{r_t}{p_{it}})}{dP_t} = \frac{dF_{ik}(\cdot)}{dP_t} = \frac{dF_{ik}(\cdot)}{d(\frac{k_{it}}{l_{it}})} \cdot \frac{d(\frac{k_{it}}{l_{it}})}{dP_t} \begin{cases} < 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}}; \\ > 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}. \end{cases} \tag{1.6.9}$$

(1.6.8) and (1.6.9) are nothing but the Stolper-Samuelson Theorem.

Q.E.D.

Proposition 1.6.3. *(The Long-run Stolper-Samuelson Theorem). Assume (A1.1)-(A1.6). A change in period t commodity price has no effect on the future returns to capital and labor.*

Proof:

$$\frac{\partial(\frac{k_{it+1}}{l_{it+1}})}{\partial P_t} = \frac{\partial(\frac{k_{it+1}}{l_{it+1}})}{\partial K_{t+1}} \cdot \frac{\partial K_{t+1}}{\partial P_t} = 0 \tag{1.6.10}$$

Q.E.D.

Intuitively, as we have seen from the Factor Price Equalization Theorem, the real return to capital and labor in one period is solely determined by the commodity prices prevailing in that period. Hence the period $t + 1$ real returns to labor and capital are affected by P_{t+1} but not P_t.

The standard Heckscher-Ohlin Theorem offers a prediction on the trade pattern between two countries with identical, homothetic tastes. In proving the analogue of the Heckscher-Ohlin Theorem in this model, we consider two countries, Country N and Country S. The two countries have the same preferences and technology as specified in Section 1.2. Furthermore, we assume

(A1.7) $U(\cdot)$ and $G(\cdot)$ are homogeneous of degree γ ($\gamma < 1$).

Proposition 1.6.4. *(The Generalized Heckscher-Ohlin Theorem). Assume (A1.1)-(A1.7). Consider two countries which are otherwise identical except the relative factor proportions measured in terms of the capital/time ($\frac{K}{H}$) ratio. In any period, the relatively capital abundant country will export the relatively capital intensive good while the relatively labor abundant country will export the relatively labor intensive good.*

Proof: Denote all the variables of Country S by adding a superscript "*". Without any loss of generality, assume that in period t Country N is capital abundant relative to Country S, ie, $\frac{K_t}{H} > \frac{K_t^*}{H^*}$.

First, consider the case where $K_t = K_t^*$. In this case $H < H^*$ by assumption. Define $\tilde{H} \equiv \frac{H+H^*}{2}$. Then $H < \tilde{H} < H^*$.

Consider a hypothetical economy with endowments K_t and $\tilde{H}$. Assume that this economy has the same preferences and production technology as Country N and Country S.

Suppose in period t, this economy is in autarky at the price $\tilde{P}_t$ and the vector

$$\tilde{R}_t' \equiv (\tilde{C}_t, \tilde{Z}_t, \tilde{I}_t, \tilde{K}_{t+1}, \tilde{k}_{1t}, \tilde{l}_{1t}, \tilde{k}_{2t}, \tilde{l}_{2t})$$

represents the solution to this economy's autarkic optimization problem in period t. In other words, given that $P_t = \tilde{P}_t$, $\tilde{R}_t$ satisfies the period t efficiency conditions (1.3.9)-(1.3.12), the resource constraints (1.3.3)-(1.3.4) and the autarkic market clearing conditions

$$\tilde{C}_t - F_1(\tilde{k}_{1t}, \tilde{l}_{1t}) = 0; \tag{1.6.11}$$

$$\tilde{I}_t - F_2(\tilde{k}_{2t}, \tilde{l}_{2t}) = 0; \tag{1.6.12}$$

Using the period t version of equation (1.4.10) and (1.4.11), one obtains

$$\frac{\partial(F_1(k_{1t}, l_{1t}) - C_t)}{\partial H}\Big|_{H=\tilde{H}} \begin{cases} > 0, & \text{if } \frac{\tilde{k}_{1t}}{\tilde{l}_{1t}} < \frac{\tilde{k}_{2t}}{\tilde{l}_{2t}}; \\ < 0, & \text{if } \frac{\tilde{k}_{1t}}{\tilde{l}_{1t}} > \frac{\tilde{k}_{2t}}{\tilde{l}_{2t}}. \end{cases} \tag{1.6.13}$$

$$\frac{\partial(F_2(k_{2t}, l_{2t}) - I_t)}{\partial H}\Big|_{H=\tilde{H}} \begin{cases} > 0, & \text{if } \frac{\tilde{k}_{1t}}{\tilde{l}_{1t}} > \frac{\tilde{k}_{2t}}{\tilde{l}_{2t}}; \\ < 0, & \text{if } \frac{\tilde{k}_{1t}}{\tilde{l}_{1t}} < \frac{\tilde{k}_{2t}}{\tilde{l}_{2t}}. \end{cases} \tag{1.6.14}$$

Since $H < \tilde{H} < H^*$, with the relative price being $\tilde{P}_t$, Country N will experience an excess demand for the labor intensive good and an excess supply of the capital intensive good while Country S will experience the opposite in period t. Therefore, if Country N and Country S are opened up for trade in period t, Country N will export the capital intensive good while Country S will export the labor intensive good.

Next, suppose $K_t \neq K_t^*$. Define $\lambda = \frac{K_t}{K_t^*}$. Then $K_t = \lambda K_t^*$ and $H < \lambda H^*$. From the above discussion we know that at $P_t = \tilde{P}_t$, a country with an endowment pair $(\lambda K_t^*, \lambda H^*)$ will experience an excess demand for the capital intensive good and an excess supply of the labor intensive good. By the homogeneity of utility function and production functions, it must be the case that, given $P_t = \tilde{P}_t$, Country S will also have an excess demand for the capital intensive good and an excess supply of the labor intensive good. Hence Country N will export the capital intensive good while Country S will export the labor intensive good if the two countries are opened up for trade in period t.

Q.E.D.

A simple way to intuitively confirm the Generalized Heckscher-Ohlin Theorem is by using equation (1.4.30) and the standard Heckscher-Ohlin Theorem. Consider two countries with the same level of capital stocks but different levels of time endowments, ie, $K_t = K_t^*$ but $H < H^*$. (1.4.30) implies that $\frac{L_t}{K_t} < \frac{L_t^*}{K_t^*}$. Then the Generalized Heckscher-Ohlin Theorem follows from the standard Heckscher-Ohlin Theorem.

Comments: Martin (1976) studied a Heckscher-Ohlin Model with variable factor supply. It is shown in his paper that the standard Heckscher-Ohlin Theorem can be restated with the term "factor endowment" being replaced by "factor usage" *if* the elasticity of labor supply is positive. If the labor supply curve is backward-bending, Martin argues, the Heckscher-Ohlin Theorem may break down, in which case a country could export the labor intensive good at one terms of trade then switch to exporting the capital intensive good at a more favorable terms of trade.

The Generalized Heckscher-Ohlin Theorem presented in this paper differs from Martin's result in two aspects. First, the Generalized Heckscher-Ohlin Theorem is stated in terms of the period t's endowments of capital and time rather than in terms of the factors that are utilized in production (ie, factor usage). Secondly, the Generalized Heckscher-Ohlin Theorem holds irrespective of the sign of the labor supply elasticity. The proof of Proposition 1.6.4 does not require a positive labor supply elasticity.

One might wonder whether the restriction that the utility function is separable in consumption and leisure in this model has played the role of ensuring a positive labor supply elasticity. The answer is "No." To see this, consider the following static consumer's optimization problem with a more general utility function:

$$\max_{C,Z,L} U(C,Z) \tag{1.6.15}$$

subject to

$$C = wL \tag{1.6.16}$$

$$Z + L = H \tag{1.6.17}$$

where $U(C, Z)$ is concave. (1.6.15)–(1.6.17) is a simple optimization problem where labor is the consumer's only source of income. Performing comparative statics on the first-order conditions of (1.6.15)-(1.6.17), one can show that

$$\frac{dL}{dw} = \frac{U_c + wLU_{cc} - U_{zc}L}{-w^2U_{cc} + 2wU_{cz} - U_{zz}}. \tag{1.6.18}$$

Proposition 1.6.4 is proven under the assumption that $U_{cz} = 0$. It is clear from equation (1.6.18) that in the case where $U_{cz} = 0$, the labor supply curve is backward-bending if $\frac{-CU_{cc}}{U_c} > 1$. Therefore, the Generalized Heckscher-Ohlin Theorem is valid even in the presence of backward-bending labor supply curve.

The comparative statics on the first-order conditions of (1.6.15)–(1.6.17) also reveals that

$$\frac{dL}{dH} = \frac{wU_{cz} - U_{zz}}{-w^2U_{cc} + 2wU_{cz} - U_{zz}} > 0 \text{ if } U_{cz} \geq 0. \tag{1.6.19}$$

ie, other things being equal, a country with a larger time endowment has a larger labor supply than a country with a smaller time endowment does if $U_{cz} \geq 0$. Therefore, it is our belief that the Generalized Heckscher-Ohlin Theorem holds with a more general utility function $U(C, Z)$ as long as $U_{cz} \geq 0$. A formal proof of this conjecture requires the re-computation of the equilibrium in the model with the utility function being $\sum_{t=0}^{\infty} \beta^t U(C_t, Z_t)$. Here we just provide an intuitive argument. With the new utility function, the first-order condition (1.3.10) now becomes

$$U_c(C_t, Z_t)F_{1l}(k_{1t}, l_{1t}) = U_z(C_t, Z_t) \tag{1.6.20}$$

Comparing two countries with the same level of time endowments but different levels of capital endowments, the country with a larger capital stock is richer and hence is expected to have a higher consumption level in equilibrium than the other country is. $U_{cz} \geq 0$ implies that the country with a higher level of consumption will also have a higher level of leisure than the other country does. As a result, the labor supply in the capital abundant country is smaller than that in the other country. Hence, by the standard Heckscher-Ohlin Theorem, it should be true that the country with a higher $\frac{K}{H}$ ratio exports the capital intensive good while the other country exports the labor intensive good.

In the presence of a backward-bending labor supply curve, a country (named as country 1 in Martin's paper) that exports the labor intensive good may reduce the output of the labor intensive good and increase the output of the capital intensive good when the terms of trade shift in favor of the labor intensive good. As a result, Martin argues, if we assume that the country that exports the capital intensive good (country 2) has the fixed factor supplies, then country 1 could export the labor intensive good at one terms of trade and then switch to exporting the capital intensive good at a more favorable terms of trade. Martin's arguments crucially depend on the assumption that country 2 has fixed factor supplies while country 1 has variable factor supplies. If we allow both countries have endogenous labor supply and have the same utility function $U(C, Z)$, then if the labor supply in country 1 is reduced as a result of the terms of trade shift, the labor supply in country 2 will be reduced as well. Under the assumption that $U_{cz} \geq 0$, country 1 still has a larger $\frac{K}{L}$ ratio than country 2 does after the terms of trade shift. The Heckscher-Ohlin Theorem still holds even in the presence of a backward-bending labor supply curve. Therefore, the breakdown of the Heckscher-Ohlin Theorem in Martin's model is due to the departure from the basic assumption of the Heckscher-Ohlin model that the two countries are identical in all aspects except factor proportions The Heckscher-Ohlin Theorem is not sensitive to the introduction of endogenous factor supplies.

Footnote

(1) Stiglitz (1970) showed that specialization in production will occur in the long-run if the time discount factors among different countries are different. However, in Chapter 2 of this thesis it is demonstrated that diversified production can be maintained in the long-run if countries have identical time discount factors. Hence the assumption (A1.5) is internally consistent if we consider a country that has the same time discount factor as that of the rest of the world.

CHAPTER 2

Long-run Equilibria in A Dynamic Heckscher-Ohlin Model

2.1. Introduction

Since the sixties, the standard Heckscher-Ohlin model of international trade has been extended to analyze problems related to the long-run equilibria in open economies (Oniki and Uzawa, 1965; Bardhan, 1965; Findlay, 1970; Stiglitz, 1970; Vanek, 1971; Bertrand, 1975; Smith, 1977; Manning and Markusen, 1982). One major issue that has been studied in these papers is the determinant of comparative advantage in the long-run (for example, Oniki and Uzawa, 1965; Findlay, 1970; Stiglitz, 1970). It appears to be the general consensus in this body of literature that the main determinant of long-run comparative advantage is the countries' savings rates(1). The question that what has caused the difference in savings rates among countries, however, is rarely explicitly discussed in the literature. The models that did endogenize savings rates (for example, Stiglitz, 1970) attributed the difference in savings rates and hence long-run comparative advantage to a difference in preferences; in particular, a difference in agents' time discount factors among countries. Yet explaining trade in terms of differences in preferences is no longer in the spirit of the Heckscher-Ohlin model in which trade arises because of differences in relative factor proportions.

There have been few attempts in the literature to explain long-run comparative advantage in terms of differences in initial factor endowment ratios among countries. This is probably because it seems intuitively obvious that in the long-run no trade would occur between two countries which are different only in their initial factor endowment ratios. Since capital stock can be accumulated in the long-run, one might expect that the two countries with identical preferences and technology would have the same capital/labor ratio in the long-run if they start with different capital/labor ratios. Therefore, while trade may occur between these two countries in some initial periods when their capital/labor ratios are different, one might expect that the two countries would eventually cease to trade with each other in the long-run. Hence in most of the existing dynamic Heckscher-Ohlin models, "long-run comparative advantage ... has nothing to do with initial endowments of capital and labor." (Stiglitz, 1970 p463.)

The goal of this chapter is to study long-run equilibria in open economies using a dynamic two-sector general equilibrium model with endogenous savings and labor supply. It is shown that, ignoring the initial conditions, there is an infinite number of possible steady states in a two-country world in which both countries have identical preferences and technology. The characteristics of these steady states are studied. With the assistance of the results by Deneckere and Pelikan (1986), it is demonstrated that given the initial endowments of the two countries, the world economy approaches a unique steady state in the long-run provided that consumers' time discount factor is sufficiently close to one.

The main contribution of this chapter is that it offers an analysis on the determinant of long-run trade pattern for the case where both countries have identical time discount factors. It is shown that the world will be in autarky in all periods if the two countries start with identical factor proportions. However, trade will occur and continue in the long-run if the initial factor proportions of the two countries are different. The country that is relatively capital abundant in the initial period remains relatively capital abundant and exports the relatively capital intensive good, while the other country remains relatively labor abundant and exports the relatively labor intensive good in the long-run. Therefore, the model offers a Heckscher-Ohlin type explanation for the long-run trade between two countries with identical preferences, that is, a difference in the initial factor proportions causes trade to continue in the long-run. We name this result the Long-run Heckscher-Ohlin Theorem.

The Long-run Heckscher-Ohlin Theorem may seem surprising at first sight. The key to understanding this result is to realize that the two countries with identical preferences but different initial capital/labor ratios will not converge to a common capital/labor ratio in the long-run. This is because in a world in which factor prices are equalized, the two countries trading with each other face the same rate of return to capital. When agents in one country find that the return on investment is high enough to increase their capital stocks, so do the agents in another country. The levels of capital stocks in the two countries move in the same direction over time. Therefore, the initial relative factor abundance and trade pattern between the two countries are maintained in the long-run.

From the perspective of the neoclassical growth theory, the Long-run Heckscher-Ohlin Theorem is an interesting result because it offers an explanation to the observed difference in per capita income levels across countries. The model in this chapter is an open economy version of the standard neoclassical growth model with endogenous labor supply. One well-known result of the existing neoclassical growth models is that in the long-run, all countries with identical preferences will always converge to the same steady state, independent of initial conditions. Yet in the real world, we observe enormous differences in per capita income levels across countries. The neoclassical growth model has been criticized for failing to account for this stylized fact (for example, Lucas, 1988). What is demonstrated in this chapter is that once international trade is incorporated into a neoclassical growth model, countries with different initial per capita income levels will no longer converge to the same steady state. The difference in the initial income levels across countries persists in the long-run.

Another interesting result of the model is that of "hysteresis," the persistence of the effects of a shock to a country's capital stock. The long-run equilibria in this model exhibit the property of economic hysteresis when the initial endowment ratios in the two countries are different. Suppose that the time discount factor is close to unity so that the world economy always approaches a steady state after a shock has occurred. While the world economy as a whole has a unique steady state level of capital stock, the distributions of the capital stock among the two countries are different before and after a shock. Hence from each individual country's perspective the shock moves its capital stock level from one steady state to another steady state[2].

This model also offers a new explanation for the difference in savings rates among countries. It is shown that two open economies with identical preferences but exporting different goods will have different average propensities to save in the steady state. The country that exports the capital intensive good has a higher steady state average propensity to save than the country that exports the labor intensive good. Therefore, while trade in the long-run is still associated with a difference in savings rates among countries, this difference is not caused by a difference in preferences. Rather, it is caused by a difference in the initial factor proportions.

This chapter is organized as follows. The basic physical environment of the world economy is outlined in Section 2.2. A dynamic model in which a social planner in an open economy maximizes social welfare over an infinite time horizon is presented in Section 2.3. Section 2.4 studies the autarkic equilibrium while Section 2.5 analyzes the properties of trading equilibria of the world economy. Some concluding remarks are presented in Section 2.6.

2.2. Economic Environment

In this section, the production and consumption side of the model are specified. Roughly speaking, the model has the standard two-by-two-by-two structure. There are two countries with identical preferences and technology. Two goods can produced using two factors: capital and labor.

The formal specification of the model follows.

Time, denoted by t, is discrete and the horizon is infinite: $t \in \{0, 1, ...\}$.

The world consists of two countries: Country N and Country S. Each country is endowed with a fixed amount of time endowment in each period, denoted by H for Country N and H^* for Country S. Time endowment is not tradeable among countries. H (and, respectively, H^*) is a measure of the size of labor force in Country N (and, respectively, Country S). Hence the per capita value of a variable x for Country N (Country S) is defined as $\frac{x}{H}$ ($\frac{x}{H^*}$). In period 0, Country N is endowed with capital stock K_0 while Country S is endowed with capital stock K_0^*. The two countries are assumed to have identical technology and identical preferences. Furthermore, it is assumed that there is no borrowing and lending between the two countries(3). In what follows, the production and consumption side of the model is specified for Country N. The variables of Country S, which will be denoted by attaching a superscript "*", can be specified in the same way.

On the *production side* of the world, two goods are produced: a consumption good and a capital good, with quantities being denoted by y_1 and y_2 for N. Both goods are tradeable and can be transported at zero cost. p_{it} ($i = 1, 2$) denotes the world price of good i in period t. Consumption good is non-storeable. In each period capital depreciates at a rate $\delta \in (0, 1]$. The depreciation rates are the same for both countries, ie, $\delta = \delta^*$.

There is a fixed continuum of firms in each industry in each country. Hence both industries are perfectly competitive. The two goods can be produced according to the following production functions: $y_{it} = F_i(k_{it}, l_{it})$ $(i = 1, 2)$, with k_{it} denoting the capital input into good i and l_{it} the labor devoted to the production of good i in period t. The nonnegative production functions satisfy:

(A2.1) $F_i(k_{it}, l_{it})$ is homogenous of degree one. $F_{ik} > 0$, $F_{il} > 0$, $F_i(0,0) = 0$, $F_{ik}(0, l_{it}) = +\infty$, $F_{il}(k_{it}, 0) = +\infty$. $F_{ikk} < 0$, $F_{ill} < 0$.

In this model, a good, say good i, is defined to be more capital intensive relative to good j if $\frac{k_i}{l_i} > \frac{k_j}{l_j}$ for any equilibrium wage-rental ratio. A standard assumption of the Heckscher-Ohlin model which will be maintained for some of the results in this chapter is that

(A2.2) there is no factor intensity reversal in production technology.

The total amount of capital stock in period t in Country N is denoted by K_t. Capital is perfectly mobile between sectors. The investment made in period t, denoted by I_t, will become productive in period $t+1$. Hence the law of motion of capital is $K_{t+1} = (1-\delta)K_t + I_t$. It is assumed that

(A2.3) there exists $\bar{K} > 0$ such that for any $k_{2t} < K_t$ and $l_{2t} < H$, $F_2(k_{2t}, l_{2t}) < \delta K_t$ if $K_t > \bar{K}$; and there exist some $k_{2t} < K_t$ and $l_{2t} < H$ such that $F_2(k_{2t}, l_{2t}) > \delta K_t$ if $K_t < \bar{K}$.

(A2.3) states that it is not possible for a self-sufficient economy to maintain capital stocks above $\bar{K}$.

Another standard assumption of the Heckscher-Ohlin model is that

(A2.4) both countries produce both goods[4].

The *consumer side* of the economy comprises a fixed continuum of identical consumers. A representative consumer in country N owns the country's intial capital stock K_0 and time endowment H. In each period he divides H into leisure and labor. His preference over consumption of the good and leisure is given by the utility function: $\sum_{t=0}^{\infty} \beta^t [U(C_t) + G(Z_t)]$, where $\beta \in (0,1)$ is the consumer's time discount factor, and C_t and Z_t denote his period t consumption of good 1 and leisure, respectively.

(A2.5) Both $U(\cdot)$ and $G(\cdot)$ are continuous, increasing, bounded below, and strictly concave functions with $U'(0) = +\infty$ and $G'(0) = +\infty$.

The standard Heckscher-Ohlin Theorem is proved under the assumption that the utility function is homothetic. In this model, the following assumption ensures the homotheticity of the utility function $\sum_{t=0}^{\infty} \beta^t [U(C_t) + G(Z_t)]$.

(A2.6) $U(\cdot)$ and $G(\cdot)$ are homogeneous of degree γ ($\gamma < 1$).

Firms and consumers are endowed with perfect foresight when making their decisions.

2.3. Social Planner's Problem

In this section, the problem faced by a social planner in an open economy is studied. The planner's problem is presented in terms of variables of Country N. It is clear that all the equations hold for Country S when the choice variables are attached with "*".

Consider a social planner who seeks to maximize the representative consumer's utility over an infinite time horizon by choosing appropriate levels of aggregate consumption, investment, labor efforts, leisure and outputs. The social planner is a price-taker in the world commodity market. The social planner's problem can be written as

$$\max_{\{C_t, Z_t, I_t, k_{1t}, l_{1t}, k_{2t}, l_{2t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [U(C_t) + G(Z_t)]; \tag{2.3.1}$$

subject to

$$p_{1t} C_t + p_{2t} I_t = p_{1t} F_1(k_{1t}, l_{1t}) + p_{2t} F_2(k_{2t}, l_{2t}) \tag{2.3.2}$$

$$K_t = k_{1t} + k_{2t} \tag{2.3.3}$$

$$H = l_{1t} + l_{2t} + Z_t \tag{2.3.4}$$

$$K_{t+1} = I_t + (1 - \delta) K_t \tag{2.3.5}$$

and $0 \leq K_0 \leq \bar{K}$.

Equation (2.3.2) distinguishes this open economy model from a standard closed economy model. Instead of demand equalling supply for both commodities, it requires that the value of consumption and investment equal the value of outputs of both goods in each period. (2.3.3) and (2.3.4) represent the full employment conditions of capital and labor in period t. In particular, (2.3.4) states that the total time available in each period is divided into working and leisure. (2.3.5) is the law of motion of capital stock.

Let $P_t \equiv \frac{p_{2t}}{p_{1t}}$ be the relative price of good 2 in terms of good 1 in period t. Assume that the world equilibrium relative prices are positive finite numbers in all periods, ie, $0 < P_t < \infty$ $(t = 0, 1, \cdots)$[5].

Define

$$T(K_t, K_{t+1}; P_t) \equiv \max_{C_t, Z_t, I_t, k_{1t}, l_{1t}, k_{2t}, l_{2t}} \{U(C_t) + G(Z_t) \mid (2.3.2), (2.3.3), (2.3.4), (2.3.5)\}. \tag{2.3.6}$$

It can be verified that $T(K_t, K_{t+1}, P_t)$ is continuous and concave in (K_t, K_{t+1}) by assumptions (A2.1) and (A2.5). Furthermore, the range of $T(K_t, K_{t+1}; P_t)$ is bounded due to assumptions (A2.3), (A2.5) and that $0 < P_t < \infty$.

Define

$$V(K_t; \{P_s\}_{s=t}^{\infty}) \equiv \max_{\{K_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} T(K_s, K_{s+1}; P_s), \quad t = 0, 1, 2, \cdots \tag{2.3.7}$$

By applying the result in Bertsekas (1976), we know that there is a unique bounded solution to the following Bellman's equation:

$$V(K_t; \{P_s\}_{s=t}^{\infty}) = \max_{K_{t+1}} T(K_t, K_{t+1}; P_t) + \beta V(K_{t+1}; \{P_{s+1}\}_{s=t}^{\infty}). \tag{2.3.8}$$

It can be shown that $V(K_t; \{P_s\}_{s=t}^{\infty})$ is concave in K_t. Furthermore, the solution to (2.3.8), combined with the solution to (2.3.6), solves the social planner's problem (2.3.1)–(2.3.5).

The solution to the social planner's problem is characterized by the following four efficiency conditions:

$$
\begin{aligned}
U_c(C_t)P_t &= \beta V_k(K_{t+1};\{P_s\}_{s=t+1}^{\infty}) \\
&= \beta U_c(C_{t+1})[F_{1k}(K_{t+1} - k_{2t+1}; l_{1t+1}) + P_{t+1}(1-\delta)]
\end{aligned}
\tag{2.3.9}
$$

$$U_c(C_t)F_{1l}(K_t - k_{2t}, l_{1t}) = G_z(H - l_{1t} - l_{2t}) \tag{2.3.10}$$

$$F_{1k}(K_t - k_{2t}; l_{1t}) = P_t F_{2k}(k_{2t}, l_{2t}) \tag{2.3.11}$$

$$F_{1l}(K_t - k_{2t}; l_{1t}) = P_t F_{2l}(k_{2t}; l_{2t}) \tag{2.3.12}$$

Since $C_t = F_1(K_t - k_{2t}; l_{1t}) + P_t F_2(k_{2t}; l_{2t}) + P_t(1-\delta)K_t - P_t K_{t+1}$, given K_t, (2.3.9)-(2.3.12) are four equations in four unknowns, K_{t+1}, l_{1t}, k_{2t}, l_{2t}. Equation (2.3.9) is a standard optimality condition governing investment. It states that the current utility loss due to an extra unit of investment must equal the discounted future utility obtained from this unit of investment. (2.3.10) illustrates that the marginal utility loss of working must equal the marginal benefit of working represented by the marginal product of labor. (2.3.11) and (2.3.12) are the familiar efficiency conditions for capital market and labor market, respectively.

By Prescott and Mehra (1980), it can be established that the solution to the above social planner's problem can be supported as a competitive equilibrium in this economy of homogeneous households and firms[6].

It is demonstrated in Chapter 1 that the Factor Price Equalization Theorem and the Stolper-Samuelson Theorem hold in this economy. Furthermore, it is shown that in any period t, in a two-country world where the two countries are otherwise identical except the factor proportions measured in terms of the capital/time ratio ($\frac{K}{H}$), the country with relatively high $\frac{K}{H}$ ratio will export the relatively capital intensive good while the other country will export the relatively labor intensive good (*The Generalized Heckscher-Ohlin Theorem*).

2.4. Autarkic Equilibrium

In this section, the autarkic equilibrium of the world economy is analysed. It is shown that there will be no trade between the two countries if they are in the steady state equilibrium before trade. Furthermore, it is demonstrated that the long-run equilibrium of the world economy is an autarkic one if the intial endowment ratios of the two countries are the same.

The efficiency conditions for an autarkic equilibrium can be obtained by imposing the autarkic market clearing conditions $C_t = F_1(t)$ and $I_t = F_2(t)$ on the efficiency conditions (2.3.9)-(2.3.12):

$$\begin{aligned} &U_c[F_1(K_t - k_{2t}, l_{1t})]P_t \\ &= \beta U_c[F_1(K_{t+1} - k_{2t+1}, l_{1t+1})][F_{1k}(K_{t+1} - k_{2t+1}, l_{1t+1}) + P_{t+1}(1-\delta)] \end{aligned} \tag{2.4.1}$$

$$U_c[F_1(K_t - k_{2t}, l_{1t})]F_{1l}(K_t - k_{2t}, l_{1t}) = G_z(H - l_{1t} - l_{2t}) \tag{2.4.2}$$

$$F_{1k}(K_t - k_{2t}, l_{1t}) = P_t F_{2k}(k_{2t}, l_{2t}) \tag{2.4.3}$$

$$F_{1l}(K_t - k_{2t}, l_{1t}) = P_t F_{2l}(k_{2t}, l_{2t}) \tag{2.4.4}$$

In a closed economy, the relative price is determined domestically. Eliminating P_t using (2.4.3), we can reduce the system of the efficiency conditions into the following three equations:

$$\begin{aligned} &U_c[F_1(K_t - k_{2t}, l_{1t})]\frac{F_{1k}(K_t - k_{2t}, l_{1t})}{F_{2k}(k_{2t}, l_{2t})} \\ &= \beta U_c[F_1(K_{t+1} - k_{2t+1}, l_{1t+1})]F_{1k}(K_{t+1} - k_{2t+1}, l_{1t+1})(1 + \frac{1-\delta}{F_{2k}(k_{2t+1}, l_{2t+1})}) \end{aligned} \tag{2.4.5}$$

$$U_c[F_1(K_t - k_{2t}, l_{1t})]F_{1l}(K_t - k_{2t}, l_{1t}) = G_z(H - l_{1t} - l_{2t}) \tag{2.4.6}$$

$$F_{1l}(K_t - k_{2t}, l_{1t})F_{2k}(k_{2t}, l_{2t}) = F_{1k}(K_t - k_{2t}, l_{1t})F_{2l}(k_{2t}, l_{2t}) \tag{2.4.7}$$

Of the special interest in the dynamic trade theory is the case of balanced growth in which capital, output, consumption and labor grow at a constant relative rate. In this model, the time endowment of each economy is assumed to be constant over time. Hence an economy is on the balanced "growth" path if the economy's capital stock and other aggregate variables equal their steady state values.

Notice that the closed economy version of this model is a standard two-sector neoclassical growth model with endogenous labor supply. It is well-known that the standard neoclassical growth model has a unique steady state. For completeness, we present the proof of this result for this model.

Lemma 2.4.1. *Assume (A2.1), (A2.3) and (A2.5). In a closed economy, there exists a unique steady state equilibrium.*

Proof: In an autarkic steady state $F_2(k_2, l_2) = I = \delta K > 0$. (A2.5) implies that the steady state consumption is positive. Therefore, both commodities are produced in an autarkic steady state.

In the steady state, (2.4.5)-(2.4.7) are reduced to:

$$\beta[F_{2k}(k_2, l_2) + (1 - \delta)] = 1 \tag{2.4.8}$$

$$U_c[F_1(K - k_2, l_1)]F_{1l}(K - k_2, l_1) = G_z(H - l_1 - l_2) \tag{2.4.9}$$

$$F_{1l}(K - k_2, l_1)]F_{2k}(k_2, l_2) = F_{1k}(K - k_2, l_1)F_{2l}(k_2, l_2) \tag{2.4.10}$$

Equations (2.4.8)-(2.4.10) are three equations in three unknowns. Straightforward calculation shows that the Jacobian associated with this system of equations does not equal zero. (In fact it is negative). Therefore, by the Implicit Function Theorem, there exists a unique solution to this equation system.

Q.E.D.

One implication of Lemma 2.4.1 is

Proposition 2.4.2. *Assume (A2.1), (A2.3), and (A2.5)-(A2.6). No trade will occur between two countries with identical preferences and technology if the two economies are in the steady state before trade.*

Proof: Define $\lambda = \frac{H}{H^*}$. Let $\bar{k}_2$, $\bar{l}_1$, $\bar{l}_2$ be the steady state values of the choice variables for Country N. Then they must satisfy conditions (2.4.8)-(2.4.10). Let

$$x^* = \frac{x}{} \tag{2.4.11}$$

where $x = \bar{k}_2, \bar{l}_1, \bar{l}_2$. By the homogeneity of the utility function and the production functions, $(\bar{k}_2^*, \bar{l}_1^*, \bar{l}_2^*)$ satisfies the steady state equilibrium conditions (2.4.8)-(2.4.10) for Country S. Furthermore, $\frac{F_{1t}(\bar{k}_1, \bar{l}_1)}{F_{2t}(\bar{k}_2, \bar{l}_2)} = \frac{F_{1t}(\bar{k}_1^*, \bar{l}_1^*)}{F_{2t}(\bar{k}_2^*, \bar{l}_2^*)}$ holds in the steady states of the two countries, which implies that the steady state autarky prices are the same in the two countries. Therefore, no trade will occur between the two countries if they are opened up for trade.

Q.E.D.

Proposition 2.4.2 is not surprising since the two countries with identical preferences have the same factor endowment ratio $(\frac{K}{H})$ in the autarkic steady state equilibrium.

One straightforward result that follows from Proposition 2.4.2 is that

Corollary 2.4.3. *Assume (A2.1), (A2.3) and (A2.5)-(A2.6). The two countries with identical preferences and technology will have the same levels of per capita income, consumption and leisure in the autarkic steady state.*

Proof: Follows from (2.4.11) and the homogeneity of the utility function and the production functions.

Q.E.D.

Proposition 2.4.4. *Assume (A2.1), (A2.3), and (A2.5)-(A2.6). If the initial endowments of the two countries are such that $\frac{K_0}{H} = \frac{K_0^*}{H^*}$, the equilibrium of the world economy is an autarkic one for all $t = 0, 1, 2, \cdots$.*

Proof: Suppose that given K_0 the sequence $\{\bar{K}_{t+1}, \bar{l}_{1t}, \bar{k}_{2t}, \bar{l}_{2t}\}_{t=0}^{\infty}$ satisfies the autarkic optimization conditions (2.4.5)-(2.4.7) for Country N. Since $\frac{K_0}{H} = \frac{K_0^*}{H^*}$, the sequence

$$\{\frac{\tilde{K}_{t+1}}{\lambda}, \frac{\tilde{l}_{1t}}{\lambda}, \frac{\tilde{k}_{2t}}{\lambda}, \frac{\tilde{l}_{2t}}{\lambda}\}_{t=0}^{\infty}$$

is feasible for country S. By the homogeneity of the utility functions and the production functions, $\{\frac{\tilde{K}_{t+1}}{\lambda}, \frac{\tilde{l}_{1t}}{\lambda}, \frac{\tilde{k}_{2t}}{\lambda}, \frac{\tilde{l}_{2t}}{\lambda}\}_{t=0}^{\infty}$ satisfies the autarkic optimization conditions for Country S. Furthermore, in each period the autarky relative prices are the same in both countries. Hence there is no need for trade between the two countries for all $t = 0, 1, 2, \cdots$.

Q.E.D.

The intuition behind Proposition 2.4.4 is clear. Since the two countries have the same initial factor proportions, they will invest in per capita terms the same amount due to the homogeneity of the utility function and production functions. As a result, the two countries have the same factor proportions in all periods. Hence they are in per capita terms identical countries in all periods.

Benhabib and Nishimura (1985) showed that the capital stocks in a standard two-sector growth model converge to either a steady state or a two-period cycle in the long-run. It is straightforward to verify that Benhabib and Nishimura's result is applicable in the closed economy version of this model.

Proposition 2.4.5. *Assume (A2.1)–(A2.3), (A2.5) and $\delta = 1$. If the initial endowments of the two countries are such that $\frac{K_0}{H} = \frac{K_0^*}{H^*}$, the time paths of the capital stocks, K_t and K_t^*, are unique. Furthermore, if the consumption good is relatively labor intensive, the capital stocks in the two countries are monotonic over time and converge to a unique steady state. If the consumption good is relatively capital intensive, the capital stocks are oscillatory over time and converge to either a steady state or a two-period cycle.*

Proof: Proposition 2.4.4 implies that we can treat the economies of N and S as two independent closed economies. Hence we can directly apply Benhabib and

Nishimura's (1985) Theorems 2, 3, 4, 5 and Remark 1 for the case where capital depreciates completely in each period.

Q.E.D.

In Proposition 2.4.5, it is assumed that $\delta = 1$. The extension of the result to the more general case $\delta \in (0, 1]$ is straightforward but involves the introduction of some new variables. Interested readers are referred to Benhabib and Nishimura (1985) for more details.

2.5. Open Economy Equilibria

This section is devoted to the analysis on the trading equilibria of the world economy. It was shown in Section 2.4 that there will be no trade between the two countries if $\frac{K_0}{H} = \frac{K_0^*}{H^*}$. Therefore, a necessary condition for a trading equilibrium to occur is $\frac{K_0}{H} \neq \frac{K_0^*}{H^*}$.

Without any loss of generality, assume that $\frac{K_0}{H} > \frac{K_0^*}{H^*}$. In other words, in period 0 Country N is relatively capital abundant while Country S is relatively labor abundant. By the Generalized Heckscher-Ohlin Theorem, Country N will export the capital intensive good while Country S will export the labor intensive good when the two countries are opened up for trade in period 0. But will trade continue in the long-run?

Proposition 2.5.1. *(The Long-run Heckscher-Ohlin Theorem). Assume (A2.1)–(A2.6). A difference in the initial factor proportions causes trade to continue in the long-run. The country that is relatively capital abundant in the initial period remains relatively capital abundant and exports the relatively capital intensive good while the other country remains relatively labor abundant and exports the relatively labor intensive good in the long-run. Furthermore, the difference in factor proportions between the two countries does not converge to zero in the long-run.*

Proof: Performing comparative statics on the system of equations (2.3.9)–

(2.3.12), we obtain that for all $t = 0, 1, 2, \cdots$,

$$\frac{d(\frac{C_t}{H})}{d(\frac{K_t}{H})} = \frac{dC_t}{dK_t} = \frac{(F_{1k} + P_t(1-\delta))\beta V_{kk}(t+1)G_{zz}}{\beta V_{kk}(t+1)G_{zz} + P_t^2 U_{cc}G_{zz} + \beta V_{kk}(t+1)F_{1l}^2 U_{cc}} > 0 \quad (2.5.1)$$

Since $\frac{K_0}{H} > \frac{K_0^*}{H^*}$ by assumption, (2.5.1) implies that $\frac{C_0}{H} > \frac{C_0^*}{H^*}$. Define $\varepsilon = \frac{C_0}{H} - \frac{C_0^*}{H^*} > 0$.

Factor price equalization implies that the two countries face the same marginal product of capital in each period. (2.3.9) and homogeneity of $U(\cdot)$ imply that for all t,

$$\frac{C_{t+1}}{C_t} = \frac{C_{t+1}^*}{C_t^*} = \left(\beta \cdot \frac{F_{1k}(t+1) + P_{t+1}(1-\delta)}{P_t}\right)^{\frac{1}{1-\gamma}} \quad (2.5.2)$$

Therefore, for all $t = 1, 2, \cdots$,

$$\frac{C_t/H}{C_t^*/H^*} = \frac{C_0/H}{C_0^*/H^*} \quad (2.5.3)$$

which implies that

$$\frac{C_t}{H} - \frac{C_t^*}{H^*} = \frac{C_t^*}{C_0^*}\varepsilon \quad (2.5.4)$$

Since $\frac{C_t}{H}$ is continuous and increasing in $\frac{K_t}{H}$, (2.5.4) implies that for all t there exists an $\eta_t > 0$ such that

$$\frac{K_t}{H} - \frac{K_t^*}{H^*} > \eta_t. \quad (2.5.5)$$

$\lim_{t\to\infty} \eta_t \neq 0$ because $\lim_{t\to\infty} \frac{C_t^*}{C_0^*} \neq 0$ in equilibrium. In other words, Country N is relatively more capital abundant than Country S in all periods, and the difference in factor proportions between the two countries does not converge to zero in the long-run.

(2.5.5) and the Generalized Heckscher-Ohlin Theorem implies that N will export the relatively capital intensive good and S will export the relatively labor intensive good in all periods.

Q.E.D.

The intuition behind the above result can be seen from equation (2.3.9). Since the two countries have identical tastes and identical production technology, the time discount factor β and the functions $U(\cdot)$ and $V(K_{t+1};\cdot)$ in (2.3.9) are the same for both countries. The two countries face the same P_t. The right-hand side of (2.3.9), $\beta V_k(K_{t+1};\cdot)$, measures the marginal benefit of an additional unit of K_{t+1} while the left-hand side, $U_c(C_t)P_t$, represents the marginal cost of an additional unit of K_{t+1}. The marginal benefit schedule of K_{t+1} is downward sloping because $V(K_{t+1};\cdot)$ is concave in K_{t+1}. Given the world relative price P_t, $F_{1l}(k_{1t},l_{1t})$ is fixed. (2.3.10) implies that in equilibrium C_t increases with Z_t, which, through the balance of payments constraint (2.3.2), ensures that C_t decreases with K_{t+1} for a given K_t. Thus, given K_t, the marginal cost schedule of K_{t+1} is upward sloping. It can be verified from (2.3.2) that, given K_{t+1}, an increase in K_t raises the country's consumption level C_t, which implies that an increase in K_t shifts the marginal cost schedule of K_{t+1} to the right. Therefore, comparing two countries with the same level of time endowments but different levels of capital stocks in period t, the country with a larger capital stock in period t will also have a larger capital stock in period $t+1$ than the other country does.

Notice that (2.5.1) and (2.5.2) implies that the capital stocks in the two countries move over time in the same direction. This is because factor prices are equalized in this model. When the agents in one country find that the rate of return on capital is high enough to increase their investment level, so do the agents in the other country.

Proposition 2.5.1 implies that if the initial pattern of relative factor abundance between the two countries is reversed due to some exogenous shocks (for example, a civil war, an earthquake), the world will reach a completely different long-run equilibrium. Hence it appears to be the case that the long-run equilibrium in this model is not unique.

Proposition 2.5.2. *Assume (A2.1), and (A2.3)-(A2.5). Ignoring the initial conditions, there is an infinite number of possible steady state equilibria in an open economy with diversified production.*

Proof: From equations (2.3.9)-(2.3.12) we obtain four efficiency equations that must be satisfied by a steady state equilibrium:

$$1 = \beta F_{2k}(k_2, l_2) + (1 - \delta) \tag{2.5.6}$$

$$U_c(C)F_{1l}(K - k_2, l_1) = G_z(H - l_1 - l_2) \tag{2.5.7}$$

$$F_{1k}(K - k_2, l_1) = P^s F_{2k}(k_2, l_2) \tag{2.5.8}$$

$$F_{1l}(K - k_2, l_1) = P^s F_{2l}(k_2, l_2) \tag{2.5.9}$$

where $C = F_1(K - k_2, l_1) - P^s F_2(k_2, l_2) - P^s \delta K$ and P^s denotes the steady state relative price.

Given P^s, the marginal products of capital and labor are determined by equations (2.5.8)-(2.5.9). On the other hand, P^s must be such that the value of $F_{2k}(\cdot)$ satisfies (2.5.6). Hence the steady state relative price P^s must be such that (2.5.6) and (2.5.8)-(2.5.9) are consistent with each other. In this model both countries have identical time discount factors, β. (2.5.6) implies that the two countries face the same marginal product of capital, $F_{2k}(\cdot)$, in a steady state[7]. By the Stolper-Samuelson Theorem we know that the marginal product of capital, $F_{2k}(\cdot)$, is a monotonic function of the relative price P. Hence by varying P we can find a unique P^s that satisfies both (2.5.6) and (2.5.8)-(2.5.9).

From each country's perspective, given P^s, (2.5.6) is redundant. The social planner in the country can solve the marginal product terms from (2.5.8)-(2.5.9). As a result, among (2.5.6)-(2.5.9) there are only three independent equations but four unknowns, K, l_1, l_2, k_2. Therefore, given that no information on the initial endowments is provided, the steady state equilibrium in this economy is indeterminate.

Q.E.D.

Proposition 2.5.2 can be explained intuitively using the Factor Price Equalization Theorem. In an open economy with diversified production, the rate of return on investment in period t measured in terms of period t capital good, $\frac{P_{t+1}}{P_t}[F_{2k}(t+1)+(1-\delta)]$, is pinned down by the world relative prices. The agents in the economy, on the other hand, discount their future consumption at a rate of $\frac{1}{\beta}$. For simplicity, suppose that the relative prices are constant over time, ie, $P_t = P_{t+1}$. If the investment on capital offers a rate of return so high that $F_{2k}(t+1)+(1-\delta) > \frac{1}{\beta}$, the agents will consume less today but more in the future, in which case the capital stock will be increasing over time in order to support the constantly higher consumption levels. If the investment offers a rate of return that is exactly equal to the agents' time discount rate, ie, $F_{2k}(t+1)+(1-\delta) = \frac{1}{\beta}$, the agents will be indifferent between consuming today and consuming tomorrow and hence may consume an equal amount each period, in which case the initial level of capital stock will be maintained over time. This steady state level of capital stock may be very high (or very low) comparing with the steady state level of capital stock of a closed economy. But given that the return on investment is fixed at $\frac{1}{\beta}$ it does not pay the agents to change the level of capital stock in the economy.

The multiplicity of steady states distinguishes an open economy dynamic model from a standard closed economy dynamic model. In a closed economy, the rate of return on capital is determined domestically. The level of the steady state capital stock must be such that the steady state rate of return on capital clears the long-run domestic capital market. Such a level of capital stock is unique given the specifications of the model. In an open economy, however, agents will maintain their current level of savings as long as the world rate of return on capital stays at its steady state value, $\frac{1}{\beta}$. As a result, ignoring the initial conditions, the candidates for possible steady states in an open economy are not unique.

To study the characteristics of the steady state equilibria in an open economy, we add an extra restriction into the system, namely, the direction and the volume of trade.

Lemma 2.5.3. *Assume (A2.1)-(A2.5). Given the direction and the volume of trade, there exists a unique candidate for the steady state equilibrium in an open economy with diversified production.*

Proof: Define $\theta_i \equiv \frac{k_i}{l_i}$, $i = 1,2$. Given the relative price P^*, equations (2.5.8)-(2.5.9) determines the capital/labor ratios in the two industries. Therefore, θ_i's are constants in a steady state.

Let L denote the steady state labor supply. k_i and l_i $(i = 1,2)$ can then be expressed in terms of two variables, K and L, and constants θ_i:

$$k_1(K,L) = \frac{\theta_1(K - \theta_2 L)}{\theta_1 - \theta_2} \tag{2.5.10}$$

$$k_2(K,L) = \frac{\theta_2(L\theta_1 - K)}{\theta_1 - \theta_2} \tag{2.5.11}$$

$$l_1(K,L) = \frac{K - \theta_2 L}{\theta_1 - \theta_2} \tag{2.5.12}$$

$$l_2(K,L) = \frac{L\theta_1 - K}{\theta_1 - \theta_2}. \tag{2.5.13}$$

Utilizing equations (2.5.10)-(2.5.13), we can write $C = F_1(k_1,l_1) + P^* F_2(k_2,l_2) - P^*\delta K$ as a function of (K,L). Hence the system of equations (2.5.6)-(2.5.9) can be reduced to the following equation in (K,L):

$$U_c[C(K,L)]F_{1l}(\theta_1) = G_z(H - L) \tag{2.5.14}$$

Denote the net export of the capital good by X_2. Suppose X_2 is exogenously given. Then,

$$F_2[k_2(K,L), l_2(K,L)] - \delta K = X_2 \tag{2.5.15}$$

Equations (2.5.14) and (2.5.15) are two equations in two unknowns, K and L. The Jacobian associated with this system of equations is

$$J = \frac{\theta_1 F_{1l} U_{cc}(F_{1k} - \delta P^*)}{\theta_1 - \theta_2}[F_{2k}\theta_2 + F_{2l}] + (U_{cc}F_{1l}^2 + G_{zz})[F_{2k}\frac{\theta_2}{\theta_1 - \theta_2} - \frac{F_{2l}}{\theta_1 - \theta_2} + \delta]$$
$$\begin{cases} > 0, & \text{if } \theta_1 < \theta_2; \\ < 0, & \text{if } \theta_1 > \theta_2. \end{cases} \tag{2.5.16}$$

In determining the sign of equation (2.5.16), notice that $F_{2k} > \delta$ in the steady state. By the Implicit Function Theorem, there exists a unique solution to K and L for the system of equations (2.5.14)-(2.5.15).

Q.E.D.

Lemma 2.5.3 implies that, without any a *priori* restrictions on the initial endowments of the two countries, any feasible trade pattern and trade volume in a two-country world can be supported as a steady state trading equilibrium. Therefore,

Proposition 2.5.4. *Assume (A2.1)-(A2.6). The steady state trade pattern in a two-country world with diversified production in both countries is indeterminate if the initial endowments of the two countries are unknown.*

Proof: Let X_2^* denote the net export of the capital good of Country S. Choose the values of X_2 and X_2^* in such a way that $X_2 = -X_2^*$, which states that the world market for the capital good clears for such values of X_2 and X_2^*. By the Walras' Law, the consumption good market also clears given X_2 and X_2^*. Furthermore, we restrict X_2 and X_2^* in the range within which both countries produce both goods. By Lemma 2.5.3, given any such a pair of X_2 and X_2^*, there exists a unique steady state equilibrium in each country. By the construction of X_2 and X_2^*, the steady state equilibria in the two countries constitute a steady state equilibrium for the world economy.

Q.E.D.

Given that the possible production pattern in each country is indeterminate in the long-run, the next logical question is whether the world total outputs are unique in the steady state equilibria.

Proposition 2.5.5. *Assume (A2.1)-(A2.6). The steady state world aggregate capital stock and outputs are unique.*

Proof: Performing comparative statics on equations (2.5.14) and (2.5.15), we obtain

$$\frac{\partial K}{\partial X_2} = -\frac{1}{J}(U_{cc}F_{1l}^2 + G_{zz}) = -\frac{U_{cc}}{J}(F_{1l}^2 + \frac{G_{zz}}{U_{cc}})\begin{cases} > 0, & \text{if } \theta_1 < \theta_2; \\ < 0, & \text{if } \theta_1 > \theta_2. \end{cases} \tag{2.5.17}$$

$$\frac{\partial C}{\partial X_2} = -\frac{1}{J}(F_{1k} - \delta P^s)G_{zz} = -\frac{U_{cc}}{J}(F_{1k} - \delta P^s)\frac{G_{zz}}{U_{cc}}\begin{cases} > 0, & \text{if } \theta_1 < \theta_2; \\ < 0, & \text{if } \theta_1 > \theta_2. \end{cases} \tag{2.5.18}$$

From (2.5.16) we can verify that, given P^s, $\frac{U_{cc}}{J}$ is a function of $\frac{G_{zz}}{U_{cc}}$.

(2.5.7) and homogeneity of $U(\cdot)$ and $G(\cdot)$ implies that $\frac{C}{Z} = \frac{C^*}{Z^*}$ in a steady state, which, in turn, implies that $\frac{U_{cc}}{G_{zz}} = \frac{U^*_{cc}}{G^*_{zz}}$. Therefore,

$$\frac{\partial K}{\partial X_2} = \frac{\partial K^*}{\partial X_2^*} \tag{2.5.19}$$

$$\frac{\partial C}{\partial X_2} = \frac{\partial C^*}{\partial X_2^*} \tag{2.5.20}$$

If we increase X_2 by ε and decrease X_2^* by ε, the change in Country N's capital stock and consumption is exactly offset by the change in Country S' capital stock and consumption in the opposite direction, which leaves the world capital stock and consumption unchanged. The world commodity market equilibrium conditions implies that total outputs of the two goods are also independent of trade patterns.

Q.E.D.

Proposition 2.5.6. *Assume (A2.1)-(A2.6). The per capita levels of national income, consumption and leisure associated with the steady state with a larger export* *(or a smaller import) of the capital intensive good are higher than the one associated with the steady state with a smaller export (or a larger import) of the same good.*

Proof: The balance of payments constraint of Country N can be written as: $X_1 = -P^s X_2$, where X_1 denotes the export of the consumption good. Using equations (2.5.18), one obtains:

$$\frac{\partial C}{\partial X_1} = -P^s \frac{\partial C}{\partial X_2} \begin{cases} > 0, & \text{if } \theta_1 > \theta_2; \\ < 0, & \text{if } \theta_1 < \theta_2. \end{cases} \tag{2.5.21}$$

Comparative statics gives

$$\frac{\partial Z}{\partial X_2} = -\frac{1}{P^s}\frac{\partial Z}{\partial X_1} = -\frac{1}{J}F_{1l}U_{cc}(F_{1k} - \delta P^s) \begin{cases} > 0, & \text{if } \theta_1 < \theta_2; \\ < 0, & \text{if } \theta_1 > \theta_2. \end{cases} \tag{2.5.22}$$

$$\frac{\partial(y_1 + P^s y_2)}{\partial X_2} = -\frac{1}{J}(F_{1k}G_{zz} + F_{1l}^2 U_{cc}\delta P^s) \begin{cases} > 0, & \text{if } \theta_1 < \theta_2; \\ < 0, & \text{if } \theta_1 > \theta_2. \end{cases} \tag{2.5.23}$$

Q.E.D.

Proposition 2.5.7. *Assume (A2.1)-(A2.6). In a steady state, the relatively capital abundant country enjoys a higher standard of living in terms of per capita levels of income, consumption and leisure than the relatively labor abundant country does.*

Proof: To be consistent with the previous assumption $\frac{K_0}{H} > \frac{K_0^*}{H^*}$, assume that Country N is relatively capital abundant and Country S is relatively labor abundant in the steady state. By the Generalized Heckscher-Ohlin Theorem, Country N exports the capital intensive good while Country S exports the labor intensive good in the steady state. By Corollary 2.5.3 and Proposition 2.5.6, $\frac{x}{H} > \frac{x^*}{H^*}$ with $x = C, Z, y_1 + P^s y_2$.

Q.E.D.

So far it has been demonstrated that the long-run equilibrium of the world economy maintains its initial pattern of relative factor abundance and that there is an infinite number of possible steady state equilibria. The remaining question is that whether the world economy will converge to any of these steady states starting from some initial capital stocks K_0 and K_0^*.

Proposition 2.5.8. *Assume (A2.1)-(A2.6) and $\delta = 1$. Given the initial endowments of the two countries, the time path of capital stocks in each country is unique. If the consumption good is relatively labor intensive, the capital stocks in each country converge monotonically to a unique steady state. If the consumption good is relatively capital intensive, the capital stocks in each country oscillate and converge to either a steady state or a two-period cycle.*

Proof: Since factor prices are equalized in this model, we can treat the world economy as a single closed economy and apply the results obtained for the standard neoclassical two-sector growth model in Benhabib and Nishimura (1985). Hence the world capital stocks will converge monotonically to a unique steady state if the consumption good is relatively labor intensive, and will oscillate and converge to either a steady state or a two-period cycle if the consumption good is relatively capital intensive.

Since the capital stocks in the two countries always move in the same direction over time and the world capital stock in each period is unique, the time path of the capital stocks in each individual country must follow the same pattern as that of the world aggregate capital stocks.

Q.E.D.

For simplicity, Proposition 2.5.8 is presented under the assumption $\delta = 1$. Again, interested readers are referred to Benhabib and Nishimura (1985) for details on the general case $\delta \in (0, 1]$.

Deneckere and Pelikan (1986) showed that the sequence of capital stocks in a standard two-sector growth model converges to a (unique) steady state if the time discount factor of the economy is sufficiently near 1. This so-called "turnpike theorem" also holds in this model.

Proposition 2.5.9. *Assume (A2.1), and (A2.3)-(A2.5). There exists $\hat{\beta} < 1$ so that if $\beta > \hat{\beta}$, the capital stocks in the two countries will always converge to a steady state.*

Proof: It can be verified that assumptions (A.1) and (A.2) in Deneckere and Pelikan (1986) are satisfied for the world economy in this model. Hence by Lemma 3 of Deneckere and Pelikan, there exists $\hat{\beta} < 1$ so that the world economy converges to a steady state if $\beta > \hat{\beta}$. Since the steady state aggregate world capital stock is unique and capital stocks in the two countries move in the same direction over time, it must be the case that the capital stocks in the two economies converge to a steady state if $\beta > \hat{\beta}$.

Q.E.D.

Notice that Proposition 2.5.9 applies to the self-sufficient economies in Section 2.4 as well.

While attributing trade in the long-run to a difference in savings rates among countries, the existing dynamic trade literature seldom investigates what has caused the difference. Intuition suggests that a difference in savings rates may arise from a difference in preferences; in particular, a difference in agents' time discount factors among countries. In this model, define the average propensity to save $s_t \equiv \frac{P_t I_t}{y_{1t} + P_t y_{2t}}$. It can be shown that $\frac{\partial s_t}{\partial \beta} > 0$, which states that a higher β indeed leads to a higher average propensity to save.

A difference in the average propensity to save, however, is not necessarily caused by a difference in preferences among countries.

Proposition 2.5.10. *Assume (A2.1)-(A2.6). In the steady state the relatively capital abundant country has a higher average propensity to save than the relatively labor abundant country.*

Proof: We continue to assume that Country N is relatively more capital abundant than Country S in the steady state. Proposition 2.5.7 implies that $\frac{L}{H} < \frac{L^*}{H^*}$. Combining it with $\frac{K}{H} > \frac{K^*}{H^*}$ we obtain $\frac{K}{L} > \frac{K^*}{L^*}$. Since

$$s = \frac{P^s \delta K}{y_1 + P^s y_2} = \frac{\delta}{F_{2k}(\theta_2) + F_{2l}(\theta_2)\frac{L}{K}}, \tag{2.5.24}$$

it follows that $s > s^*$.

Q.E.D.

To summarize, in this section we analyzed the long-run equilibrium of the world economy under the assumption that the two countries start with different initial factor proportions. It is shown that the steady state world aggregate capital stock is unique and is independent of initial conditions. The distribution of the aggregate capital stock among the two countries, however, depends on the initial endowment ratios. The country that is relatively capital abundant in the initial period will remain relatively capital abundant while the other country will remain relatively labor abundant in the long-run. The difference in factor proportions between the two countries does not converge to zero in the long-run.

One implication of these results is that the economies in this model have the property of economic hysteresis. Suppose that the time discount factors of the two countries are sufficiently close to unity so that the world economy always converges to a steady state in the long-run. Such a steady state equilibrium is "unstable" in the sense that the capital stock in each country does not return to the same steady state level after a shock to the capital stock in a country. Instead, a shock moves the world economy to a steady state with a different distribution of the world capital stock among the two countries.

In this model, trade between two countries continues in the long-run if they start with different factor endowment ratios. As a result, the two trade partners have different savings rates in the steady state. Thus, this model offers a new explanation to the difference in savings rates among countries. In this model it is the difference in the initial factor endowment ratios that causes the difference in the steady state savings rates.

2.6. Concluding Remarks

The results which we have obtained in this chapter have clear empirical and policy implications. The results of multiplicity of possible steady states for an open economy and the differences in the per capita levels of national income, consumption, and leisure associated with these steady states can be used to explain the observed enormous diversity in per capita income and living standards among countries. The uniqueness of the steady state given the initial endowments suggests that an open economy could be stuck in an equilibrium with low per capita income level, while Proposition 2.5.6 implies the possibility that the government in such a country may be able to "push" the economy into a steady state with a higher per capita income level by means of policies that promote the production and the export of capital intensive goods. Whether such policies are welfare-improving, however, is an open question subject to investigation.

One interesting question in the context of North-South trade relations is that whether the South would catch up the North in terms of the standards of living if the South had access to the same technology as the North. This model suggests that, while the long-run equilibrium in which the North and the South share the same standards of living exists, given that the North starts with more capital stock in per capita terms than the South the world economy would approach a long-run equilibrium where the rich countries remain rich and the poor countries remain poor.

In this model the utility function is assumed to be separable in consumption and leisure. It is clear that one can re-compute the equilibrium of the model if

$U(C) - G(Z)$ is replaced by a more general function $U(C, Z)$. It can be verified that in this more general set-up, the Factor Price Equalization Theorem holds. It is our belief that, by adding an additional assumption $U_{cz} \geq 0$, one can derive the Long-run Heckscher-Ohlin Theorem from this more general set up. $U_{cz} \geq 0$ implies that in equilibrium C_t increases with Z_t, which ensures that the marginal cost schedule of K_{t+1} is upward sloping. Therefore, given the assumption $U_{cz} \geq 0$, the arguments presented in the discussion immediately after the proof of Proposition 2.5.1 is applicable in this more general set-up.

The welfare issues associated with trade and autarky are not explicitly discussed in this chapter. Proposition 2.5.7, combined with Corollary 2.4.3, implies that for the relatively labor abundant country free trade leads to a lower steady state welfare level than the one associated with the autarkic steady state. This implication, however, does not necessarily invalidates the Theorem of Gains From Trade. The welfare comparison in this case should be made in terms of the discounted sums of utilities derived from consumption and leisure over an infinite time horizon rather than the utility levels in the steady state. Therefore, one interesting topic for future research is to formally analyze the issue of gains from trade in this model.

Footnotes

(1) Population growth rate is also a determinant of long-run comparative advantage in some models.

(2) Manning and Markusen (1982) also found the result of multiple equilibria in their model. But they did not answer the question that why and how a particular equilibrium is reached. In particular, they did not have a satisfactory explanation as to why and how the world economy would approach a trading equilibrium as opposed to an autarkic equilibrium.

(3) This assumption requires that trade be balanced in each period. The goal of this chapter is to find the determinant of trade in the long-run between two countries with identical preferences. This assumption serves to suppress the issue of current account balance, an issue that is not the subject of this chapter.

(4) Stiglitz (1970) showed that specialization will occur in the steady state if the time discount factors of the two countries are different. But in this chapter we are interested in the case where both countries have the same time discount factor. In Stiglitz's model, it is the difference in the time discount factors that drives the specialization result. Once we let the time discount factors be the same for the two countries, the arguments for specialization no longer exist. (See also footnote 7.) As will become clear in Sections 2.4 and 2.5, (A2.4) can be satisfied by choosing appropriate configurations of initial endowments for the two countries.

(5) P_t will be determined endogenously in Sections 2.4 and 2.5, where the equilibria of the world economy are studied. It is clear that the assumption $0 < P_t < \infty$ made for the social planner's problem is internally consistent with the model.

(6) The conventional welfare theorem is not applicable here since this is a model with an infinite number of commodities.

(7) In Stiglitz's (1970) model the two countries is assumed to have different β's, which leads to different steady state marginal products of capital for the two countries. As a consequence, specialization in production occurs in the steady state.

CHAPTER 3

Will Economic Activities Lead to a Climatic Chaos?

3.1. Introduction

Human activities are having an increasing effect on our climate. Industrial development, the burning of fossil fuel, deforestation and even agricultural activities are changing the composition of the Earth's atmosphere. There is widespread concern that certain gases released by these activities are now building up to concentrations sufficient to affect the global climate through a process known as the "greenhouse effect." This refers to the process whereby a class of atmospheric gases (so-called greenhouse gases) contribute to warming of the Earth's environment. These gases include carbon dioxide (CO_2), chlorofluorocarbons (CFC's), ozone, and others; all are transparent to incoming radiation from the Sun and absorb outgoing radiation, re-emitting this energy in all directions. Therefore, an increase of the atmospheric concentration of greenhouse gases leads to a warming of the Earth's surface and lower atmosphere.

The greenhouse effect has received a good deal of attention in recent years. Extensive research has been conducted by atmospheric scientists to assess the problem of increased concentration of greenhouse gases (in particular, CO_2) in the atmosphere and to project future climate changes. (For a review on the methodology and results of these models, see Mintzer and Kristoferson, 1986, and Dickinson, 1983).

One problem that is common to the existing research on future climate change is the neglect of the forces of the market mechanism. The existing estimates on future climate change is conducted under the assumption of exogenous economic activities. Clearly, such an assumption suppresses the effects of the reactions of economic activities to the predicted changes in climate. For example, one of the major worries in the global warming literature is that higher global average temperature may have severe adverse effects on world agricultural production. However, if the productivity of agriculture falls as a result of the global warming, the relative price of agricultural commodities is expected to rise, which, under certain demand conditions, will drive down the production of the manufactured goods and hence the emissions of the greenhouse gases. Taking this force into account, it is probable that the existing models may have wildly over-estimated the amount of future emissions of the greenhouse gases and their impact on future climate.

Such criticisms on the computational models that ignore the functioning of market mechanism is not new in the economics literature. One of the major criticisms of Solow on the well-known the Club of Rome Report in the early 70s is that the report fails to take into account the reaction of the built-in market mechanism (Solow 1973). The Club of Rome Report predicts that the world economy will overshoot and then collapse some time in the middle of next century. However, once the market mechanism is incorporated into these Doomsday Models, Solow argues, the world economy will smoothly approach its natural limit, if there were indeed a limit of growth. Furthermore, as a resource becomes more and more scarce, the price mechanism will create strong incentives to reduce the consumption of the resource and to invent substitutes. Therefore, Solow argues, there is no reason to believe that such limit of growth must exist.

While the importance of market mechanism on the global warming problem is widely recognized by economists, so far in the literature there have been no studies that attempt to incorporate the market mechanism into the climate modelling[(1)]. It has been argued that because of the complexity of the global warming problem, the formal modelling of the dynamic interaction between economic activities and climate system is unlikely to produce major insights (See Lave, 1982). This leads to the proposal of the alternative approach of "a more informal interaction among people representing the important scientific areas": "The models of each discipline would define internally consistent 'scenarios'... These scenarios provide a structure within which scientists of diverse fields can interact" (Lave, 1982).

This chapter is the first attempt in the literature to formally model the dynamic interaction between economic activities and the climate system. In this chapter, a dynamic two-sector general equilibrium model is constructed. This chapter attempts to answer the question, "what are the possible scenarios that would arise from the interaction between economic activities and the climate system: would the world temperature reach a steady state? or would it be increasing forever? or something else?" The answer to the above question will also provide an answer to the following question. The greenhouse effect is supposedly caused by economic activities. Will the market mechanism itself be able to correct this problem? In other words, will the forces of the price mechanism be sufficient to stop the trend of global warming?

In this chapter, the model is constructed in such a way that the market has a built-in self-stabilizing mechanism that offsets rising temperature. The climate system, which is taken from the literature on climate forecast, is also stable. The interaction between the two stable systems, however, does not necessarily lead to a stable long-run equilibrium. It is shown that the characteristics of the equilibrium time path depends critically on the Earth's natural cooling tendency. The world economy and temperature will reach a steady state as long as the rate at which the Earth sheds heat is not too small. If this decay is too small, however, competitive equilibrium will, under certain conditions, lead to climatic cycles or even a climatic chaos. It is shown that under certain conditions the equilibrium law of motion of temperature displays sensitive dependence on initial conditions.

The possible existence of sensitivity in the time path of temperature casts doubts on the existing projection about future climate changes. For a dynamical system that exhibits sensitivity, a small error of measurement of the initial state may result in very large prediction errors for future dates, even if the forecaster knows very well the law of motion of the system. In the reality, of course, we cannot even claim that we know the system well.

It should be pointed out that to date there are still disagreements among scientists with regard to whether the recent warming is caused by human contributions to atmospheric CO_2[2]. In this analysis, we do not take up this debate. Rather we start from the premise that emission of greenhouse gases by human activities does lead to higher global temperature. However, the main proposition of the paper that the dynamic interaction between a stable natural system and a self-stabilizing market mechanism can, under certain conditions, lead to chaotic behavior is still a valid one even if the greenhouse effect were proven to be insignificant in the future. it is our belief that this proposition will prove its importance over time as man's capacity to affect nature grows larger and larger as a result of technological progress.

This chapter is organized as follows. Section 3.2 describes the economic and natural environment while Section 3.3 studies the competitive equilibrium. Section 3.4 analyses the properties of the law of motion of the global temperature. Sections 3.5 and 3.6 consider the equilibrium time path of temperature under different conditions based on the results obtained in Section 3.4. Section 3.7 investigates the conditions under which a climatic chaos occurs while Section 3.8 offers concluding remarks.

3.2. Economic and Natural Environment

In this section the production, consumption and climatic aspects of the model are specified. Roughly speaking, we shall consider a competitive world with no national boundaries. There are two goods in the economy, one of which is an agricultural good and the other a manufactured good. The productivity of the agricultural sector is affected by the global temperature. The manufacturing activities, on the other hand, affects temperature level. Firms take temperature levels as given. There is no market for temperature.

The formal specification of the model follows.

Time, denoted t, is discrete and the horizon is infinite: $t \in \{0, 1, ...\}$.

On the *production side* of the economy, two non-storeable goods are produced: an agricultural good and a manufactured good, with quantities being denoted by S_1 and S_2.

There is a fixed continuum of firms in each industry. Hence both industries are perfectly competitive. Labor is the only input of production. At each date, a representative firm in industry i ($i = 1, 2$) chooses the level of employment in the industry, ϕ_{it}.

The production technology of both goods exhibits constant return to scale. The productivity of labor in the manufacturing sector does not depend on climate and is denoted by b. The output of the manufacturing sector at date t can then be written as $S_{2t} = b\phi_{2t}$. In the agricultural sector, however, the productivity of labor depends on one aspect of the climate, namely the global temperature. Let $a(\tau_t)$ denote the productivity coefficient of the agricultural sector, ie, $S_{1t} = a(\tau_t)\phi_{1t}$, where τ_t is the world average temperature level in period t. It is assumed that $a(\tau_n) > 0$; $a'(\tau_t) > 0$ if $\tau_t < \bar{\tau}$ and $a'(\tau_t) < 0$ if $\tau_t > \bar{\tau}$; $a(\tau_u) = 0$; and $a''(\tau_t) < 0$. τ_n denotes the "natural" temperature level, ie, the level at which the global temperature would stay in the absence of any manufacturing activities. $\bar{\tau}$ is some critical level of temperature for the agriculture sector. $\tau_n < \bar{\tau}$. $\tau_u (> \bar{\tau})$ is the temperature level at which the agriculture productivity equals zero. Therefore, by assumption the agriculture productivity is positive when there have been no manufacturing activities. A higher level of temperature improves the agriculture productivity as long as the temperature is below the critical value $\bar{\tau}$. As temperature level exceeds $\bar{\tau}$, however, higher temperature will reduce the productivity of the agriculture sector. The agriculture productivity eventually approaches zero as temperature level reaches τ_u.

The *consumer side* of the economy comprises a fixed continuum of identical consumers. A representative consumer is endowed with one unit of labor endowment which is supplied inelastically. He has no initial wealth. His preference over consumptions of the agricultural good and the manufactured good at date t, denoted by C_{1t} and C_{2t}, respectively, is represented by the the utility function $\sum_{t=0}^{\infty} \beta^t U(C_{1t}, C_{2t})$, where $U(\cdot)$ satisfies:

(A3.1) $U(C_1, C_2) \in C^3$ and $\lim_{C_1 \to 0} U_1(C_1, C_2) = +\infty$, $\lim_{C_2 \to 0} U_2(C_1, C_2) = +\infty$.

(A3.2) $U(C_1, C_2)$ is concave and homothetic.

(A3.3) The elasticity of substitution between the two goods is less than unity.

We shall make more assumptions on preferences after we develop more notations in Section 3.4.

The law of motion of the *global average temperature* is characterised by a variation of the zero-dimensional climate system model presented in Dickinson (1986)[3]:

$$\tau_{t+1} = (1 - c)(\tau_t - \tau_n) + \tau_n + g(S_{2t}) \tag{3.2.1}$$

where $c \in (0\ 1)$, $g(S_{2t}) \in C^2$, $g(0) = 0$, $g'(S_{2t}) > 0$ and $g''(S_{2t}) \leq 0$.

(3.2.1) states that the manufacturing activities raises temperature[4]. When temperature is above its natural level, τ_n, the nature has the ability of absorbing a percentage of the excess greenhouse gases and cooling down the climate towards its natural level at a rate c. Since it is assumed that $\tau_n < \bar{\tau}$, starting from the point of time where no manufacturing activities had taken place in the past "some" manufacturing activities would be good for the production of the agriculture good by assumption. The world starts at a temperature level $\tau_0 \in [\tau_n, \tau_u)$.

As a climate system model, (3.2.1) has no independent predictive value since the parameters of the model are best obtained from more detailed models. (3.2.1) is used to interpret and summarize the results of more detailed and complex models (Dickinson, 1986). Since the goal of this chapter is to generate qualitative predictions about the outcomes of dynamic interactions between the economic system and the climate system, rather than to produce quantitative estimates of the climate system, (3.2.1) is a sufficient representation of the climate system for our purposes.

Throughout the analysis it is assumed that

$$(A3.4) \qquad g(b) < \bar{\tau} - \tau_n,$$

which states that the greenhouse effect caused by one period of manufacturing activities alone would not be sufficient to drive temperature from its natural level, τ_n, to $\bar{\tau}$ even if all the labor supply were devoted to manufacturing activities. (A3.4) can be satisfied by choosing appropriate unit for time.

Without any loss of generality we choose the unit of temperature level in such a way that $\tau_n = 0$.

3.3. Competitive Equilibrium

Let p_{1t} and p_{2t} denote the period t prices of the agriculture good and the manufactured good, respectively. Let w_t be the competitive wage rate prevailing in period t and R_t the competitive nominal interest rate between period $t-1$ and t. $R_0 = 1$ by definition. The optimization problem faced by a representative consumer in this economy, in addition to the trivial decision of inelastically supplying one unit of labor, is to maximize the sum of discounted utility obtained from consuming the manufactured good and the agriculture good taking $\{p_{1t}, p_{2t}, w_t, R_t\}_{t=0}^{\infty}$ as given. The consumer's problem can then be written as:

(C)

$$\max_{\{C_{1t}, C_{2t}, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_{1t}, C_{2t}) \tag{3.3.1}$$

subject to

$$p_{1t}C_{1t} + p_{2t}C_{2t} + A_{t+1} = w_t + R_t A_t \tag{3.3.2}$$

and $A_0 = 0$. A_t is the consumer's wealth in period t.

The first-order conditions are:

$$\frac{p_{2t}}{p_{1t}} U_1(C_{1t}, C_{2t}) = U_2(C_{1t}, C_{2t}) \tag{3.3.3}$$

$$\frac{U_1(C_{1t}, C_{2t})}{p_{1t}} = \beta U_1(C_{1t+1}, C_{2t+1}) \frac{R_{t+1}}{p_{1t+1}} \tag{3.3.4}$$

Define $\delta_t = \prod_{s=0}^{t} R_s$. Since each firm is infinitesimal, firms take temperature as given when choosing output level in each period. The optimization problems faced by a representative firm in the agricultural sector and, respectively, the manufacturing sector are:

(A)

$$\max_{\{\phi_{1t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{1}{\delta_t} [p_{1t} a(\tau_t) \phi_{1t} - w_t \phi_{1t}] \tag{3.3.5}$$

(M)

$$\max_{\{\phi_{2t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{1}{\delta_t}[p_{2t}b\phi_{2t} - w_t\phi_{2t}] \tag{3.3.6}$$

Since there is no capital good in this model, the firms' optimization problems are in effect static. The first-order conditions are:

$$p_{1t}a(\tau_t) \geq w_t; \quad \phi_{1t} \geq 0; \quad \phi_{1t}(p_{1t}a(\tau_t) - w_t) = 0 \tag{3.3.7}$$

$$p_{2t}b \geq w_t; \quad \phi_{2t} \geq 0; \quad \phi_{2t}(p_{2t}b - w_t) = 0. \tag{3.3.8}$$

Market clearing requires that

$$C_{1t} = a(\tau_t)\phi_{1t} \tag{3.3.9}$$

$$C_{2t} = b\phi_{2t} \tag{3.3.10}$$

$$\phi_{1t} + \phi_{2t} = 1 \tag{3.3.11}$$

$$A_t = 0 \tag{3.3.12}$$

Equations (3.3.9) and (3.3.10) are goods market clearing conditions. (3.3.11) is the labor market clearing condition. Since all consumers are identical, no borrowing and lending occurs among consumers in equilibrium. This is captured by (3.3.12).

We define a competitive equilibrium in this economy as follows:

Definition 3.3.1. *A competitive equilibrium is a collection of sequences* $\{p_{1t}, p_{2t}, w_t, R_t, C_{1t}, C_{2t}, \phi_{1t}, \phi_{2t}, \tau_{t+1}, A_{t+1}\}_{t=0}^{\infty}$ *and scalars* τ_0, $A_0 = 0$ *such that:*

(1) given the sequence $\{p_{1t}, p_{2t}, w_t, R_t\}_{t=0}^{\infty}$ *and scalar* A_0, $\{C_{1t}, C_{2t}, A_{t+1}\}_{t=0}^{\infty}$ *solves the consumers' problem (C).*

(2) given the sequence $\{p_{1t}, p_{2t}, w_t, R_t, \tau_t\}_{t=0}^{\infty}$, $\{\phi_{1t}\}_{t=0}^{\infty}$ *solves the agricultural sector's problem (A).*

(3) given the sequence $\{p_{1t}, p_{2t}, w_t, R_t\}_{t=0}^{\infty}$, $\{\phi_{2t}\}_{t=0}^{\infty}$ *solves the manufacturing sector's problem (M).*

(4) given $\tau_0 \in [0, \tau_u)$, $\{\tau_{t+1}\}_{t=0}^{\infty}$ evolves according to $\tau_{t+1} = (1-c)\tau_t + g(S_{2t})$. Furthermore, $\tau_t \in [0, \tau_u)$ for $t = 0, 1, 2, \cdots$.

(5) $\{C_{1t}, C_{2t}, \phi_{1t}, \phi_{2t}, A_{t+1}\}_{t=0}^{\infty}$ satisfies the market clearing conditions (3.3.9)–(3.3.12).

Since there is no borrowing and lending in equilibrium, the consumers' optimization problem in each period is essentially static. Hence the competitive equilibrium defined above can be viewed as a sequence of one-period equilibria with temperature level in each period being pre-determined by (4) of Definition 3.3.1. We can thus choose one price numeraire for each period. Define the relative price of the manufactured good, $p_t \equiv \frac{p_{2t}}{p_{1t}}$ and the real rate of interest, $r_t \equiv \frac{p_{1t-1}}{p_{1t}} R_t$.

Let $\psi_t \equiv \phi_{2t}$. (3.3.11) implies that $\phi_{1t} = 1 - \psi_t$. (3.3.7) and (3.3.8) together with (3.3.3) imply that $p_t = \frac{a(\tau_t)}{b}$. Therefore, a competitive equilibrium is characterized by the following three conditions:

$$\frac{a(\tau_t)}{b} U_1(a(\tau_t)(1-\psi_t), b\psi_t) = U_2(a(\tau_t)(1-\psi_t), b\psi_t) \tag{3.3.13}$$

$$U_1(a(\tau_t)(1-\psi_t), b\psi_t) = \beta U_1(a(\tau_{t+1})(1-\psi_{t+1}), b\psi_{t+1}) r_{t+1} \tag{3.3.14}$$

$$\tau_{t+1} = (1-c)\tau_t + g(S_{2t}) < \tau_u \tag{3.3.15}$$

Lemma 3.3.1. *Given $\tau_t \in [0, \tau_u)$, there is a unique interior solution to (3.3.13) for ψ_t.*

Proof: Note that, given $\tau_t \in [0, \tau_u)$, (3.3.13) is equivalent to the first-order condition to the following problem:

$$\max_{\psi_t} U\left(a(\tau_t)(1-\psi_t), b\psi_t\right)$$

which has a unique interior solutior to ψ_t as a function of τ_t given the concavity assumption on $U(\cdot)$.

Q.E.D.

Denote this solution to (3.3.13) by $\psi^*(\tau_t)$. By the concavity of the utility function and the implicit function theorem we know that $\psi^*(\tau_t)$ is continuous and differentiable in τ_t.

The existence of a competitive equilibrium depends on the magnitude of c. For c sufficiently close to 0 and for some $\tau_t \in [0,\ \tau_u)$ the solution to (3.3.13) violates (3.3.15), in which case there will be no interior solution to r_{t+1} in (3.3.14). We will present the exact conditions under which a competitive equilibrium exists after we develop enough notations in Section 3.3.4. Here we present a "partial" existence result.

Proposition 3.3.2. *There exists a $\bar{c} \in (0,\ 1)$ such that for all $c \in [\bar{c},\ 1]$ a unique competitive equilibrium exists for a given $\tau_0 \in [0,\ \tau_u)$.*

Proof: Pick $\bar{c} = \frac{g(b)}{\tau_u}$ (<1 by (A3.4)). Since $S_{2t} \leq b$ by the labor resource constraint and (3.3.13), for all $c \in [\bar{c},\ 1]$ $\tau_{t+1} = (1-c)\tau_t + g(S_{2t}) < \tau_u$, $\forall \tau_t \in [0,\ \tau_u)$. Therefore, given $c \in [\bar{c},\ 1]$, the unique solution to (3.3.13) will also satisfy (3.3.15). From (3.3.14) one can solve for the equilibrium r_{t+1}.

Q.E.D.

(3.3.3) and Assumption (A3.2) implies that in equilibrium the consumption ratio of the two goods is a function of the relative price of the two goods, ie,

$$\frac{C_{2t}}{C_{1t}} = \xi(p_t)$$

with $\xi(\cdot)$ being C^2. The assumptions on the utility function implies that $\xi'(p) < 0$; $\lim_{p\to 0} \xi(p) = +\infty$; $\lim_{p\to\infty} \xi(p) = 0$ and that

$$\frac{d[p\xi(p)]}{dp} = \xi(p)[1 + \frac{p\xi'(p)}{\xi(p)}] > 0 \tag{3.3.16}$$

$$\lim_{p\to 0} p\xi(p) < \infty \tag{3.3.17}$$

$$\lim_{p\to 0} \frac{d[p\xi(p)]}{dp} = \lim_{p\to 0} \xi(p)[1 + \frac{p\xi'(p)}{\xi(p)}] = +\infty \tag{3.3.18}$$

In particular, (3.3.17) is derived from the fact that $p\xi(p)$ is finite for $p \in (0, \infty)$ and is increasing in p.

The equilibrium output of the manufactured good equals

$$S_{2t} = \frac{bp_t\xi(p_t)}{1 + p_t\xi(p_t)} \tag{3.3.19}$$

with $p_t = \frac{a(\tau_t)}{b}$. Therefore,

$$\frac{dS_2}{dp} = \frac{b\frac{d[p\xi(p)]}{dp}}{(1 + p\xi(p))^2} > 0 \tag{3.3.20}$$

Remarks: The assumption (A3.3) is necessary and sufficient in determining the sign of (3.3.20). (A3.3) ensures that the income effect dominates the substitution effect when the productivity of agriculture (and hence the relative price) changes.

One goal of this chapter is to answer the question whether the market mechanism can function effectively to stop the trend of global warming. We want to study the dynamic time path of global temperature and outputs in an environment where in each period the market responds to the price signals. Under (A3.3) and concavity assumptions on $a(\tau)$ and $g(S_2)$, this model has a built-in mechanism that serves to offset the rising temperature when τ exceeds $\bar{\tau}$; when the relative price of the manufactured good falls as a result of falling agriculture productivity, the equilibrium output of the manufactured good also falls. But is this mechanism sufficient to correct the global warming problem? The rest of the chapter attempts to answer this question.

3.4. Properties of the Law of Motion of Temperature

In this section, we shall study the properties of the law of motion of the global temperature. Strictly speaking, the results to be presented in this section are the mathematical properties of a family of maps to which the equilibrium law of motion of temperature belongs. Not all the maps in this family can be supported as the outcome of a competitive equilibrium, as we shall see later in this section.

The section starts with a list of additional assumptions on preferences and on the climate system, followed by analysis on the properties of the maps. The proofs of most results in this section are straightforward but tedious. Thus they are left to Appendix II.1.

Two additional assumptions on the preferences will be maintained throughout the paper:

(A3.5) $\lim_{p\to 0} p\xi(p) = 0$;

(A3.6) $\eta_{\sigma p} \equiv \frac{p}{\sigma}\frac{d\sigma}{dp} \leq (1-\sigma)$ where σ is the elasticity of substitution in consumption.

(A3.5) implies that $\lim_{p\to 0} S_2(p) = 0$; in other words, the equilibrium output of the manufactured good approaches zero as the relative price of the manufactured good approaches zero.

(A3.6) requires that the elasticity of substitution in consumption be relatively stable as price varies.

Assumptions (A3.5) and (A3.6) does not necessarily impose more restrictions on the utility functions than (A3.1)-(A3.3). In the case of CES utility function, (A3.1)-(A3.3) implies (A3.5) and (A3.6). In fact, any CES utility function with elasticity of substitution less than unity satisfies all of (A3.1)-(A3.3) and (A3.5)-(A3.6).

To simplify the presentation it is assumed that

(A3.7) $-\tau D_\tau^2 g(\tau) > 1$ for all $\tau \in [\bar{\tau}, \tau_u]$.

(A3.7) requires that $g(S_2(\tau))$ have enough curvature in τ for τ in the interval

$[\bar{\tau},\ \tau_u]$. Notice that $D_\tau^2 g(\tau) < 0$ and that $D_\tau^2 g(\tau) \to -\infty$ as $\tau \to \tau_u$. (A3.7) is not a binding restriction for τ large enough. Imposition of (A3.7) does not change the number of possible cases in this model. It serves to restrict the occurrence of each case to one interval (as opposed to several intervals) of c values.

From (3.3.15) and (3.3.19), we can write the equilibrium time path of temperature as follows:

$$\tau_{t+1} = (1-c)\tau_t + g\left[\frac{bp_t\xi(p_t)}{1+p_t\xi(p_t)}\right] \tag{3.4.1}$$

(3.4.1) is not well defined at $\tau_t = \tau_u$ given the existing assumptions. If we define

$$f(\tau_t; c) \equiv \begin{cases} (1-c)\tau_t + g[S_{2t}(\tau_t)], & \text{if } \tau_t \in [0 \quad \tau_u); \\ (1-c)\tau_u, & \text{if } \tau_t = \tau_u, \end{cases} \tag{3.4.2}$$

then $f(\tau_t; c)$ is continuous in τ_t in the whole interval of $[0,\ \tau_u]$ since $\lim_{\tau_t \to \tau_u}\{(1-c)\tau_t + g[S_2(\tau_t)]\} = (1-c)\tau_u$. It is easy to verify that $f(\tau_t; c)$ is differentiable in τ_t in the interval $[0,\ \tau_u]$. Furthermore,

$$\frac{df}{d\tau_t} = (1-c) + \frac{dg}{d\tau_t}\begin{cases} > 1-c, & \text{if } \tau_t < \bar{\tau}. \\ < 1-c, & \text{if } \tau_t > \bar{\tau}. \end{cases} \tag{3.4.3}$$

$$\frac{d^2 f}{d\tau_t^2} = \frac{d^2 g}{d\tau_t^2} < 0 \tag{3.4.4}$$

(A3.6) and concavity assumptions on $g(\cdot)$ and $a(\cdot)$ are required in determining the sign of (3.4.4).

Notice that for $c \in [0,\ 1]$, $f(\tau_t; c)$ constitutes a family of one-dimensional continuous maps. The following is some mathematical properties of this family of maps.

Lemma 3.4.1. *Assume (A3.1)-(A3.3) and (A3.5)-(A3.6). There is a unique* $\tau_s \in (0,\ \tau_u]$ *such that* $f(\tau_s) = \tau_s$. *There is a unique* $\tau_m \in (0,\ \tau_u)$ *such that* $f(\tau_m) \geq f(\tau)$ *for all* $\tau \in [0,\ \tau_u]$.

Proof: See Appendix II.1.

Define $\tau'_m \equiv f(\tau_m)$. $\tau_s(c)$, $\tau_m(c)$ and $\tau'_m(c)$ are continuous functions of c. Lemma 3.4.1 implies that f has the shape as shown in Figure 3.4.1. Notice that since $\lim_{\tau \to \tau_u} D^2_\tau f = -\infty$, f becomes steep very fast as τ approaches τ_u.

Recall that c is the rate at which the nature cleans up the excess amount of the greenhouse gases. As c varies from 1 (complete cleaning up) to 0 (no cleaning up), f shifts up. Analysis on how f behaves as c changes will help us to determine how the existence of an equilibrium is related to the value of c.

Lemma 3.4.2. *Assume (A3.1)–(A3.3) and (A3.5)–(A3.6). As c decreases, the values of τ_s, τ_m, and τ'_m increases.*

Proof: See Appendix II.1.

Therefore, as c decreases, both the critical point and the fixed point moves towards the right, while the maximum height of f shifts up.

Lemma 3.4.3. *Assume (A3.1)–(A3.7). There exists a unique $c^1 \in (0,\ 1)$ such that $\tau_m(c) = \tau_s(c) = \tau'_m(c)$. Furthermore, $\tau_s > \tau_m$, $\tau'_m > \tau_m$ for all $c \in [0,\ c^1)$; $\tau'_m < \tau_m$, $\tau_s < \tau_m$ for all $c \in (c^1,\ 1]$.*

Proof: See Appendix II.1.

Lemma 3.4.4. *Assume (A3.1)–(A3.7). There exists a unique $c^0 \in (0,\ c^1)$ such that for all $c \in [0,\ c^0]$, $\tau'_m \geq \tau_u$ and that for all $c \in (c^0,\ c^1)$, $\tau'_m < \tau_u$.*

Proof: See Appendix II.1.

With enough notations developed, we are able to give a more precise existence result than Proposition 3.3.2.

Proposition 3.4.5. *There exists a unique competitive equilibrium for $c \in (c^0,\ 1]$.*

Proof: By the definition of c^0, we know that the interior solution to (3.3.13) satisfies (3.3.15). Hence (3.3.14) can be solved for τ_{t+1}

Q.E.D.

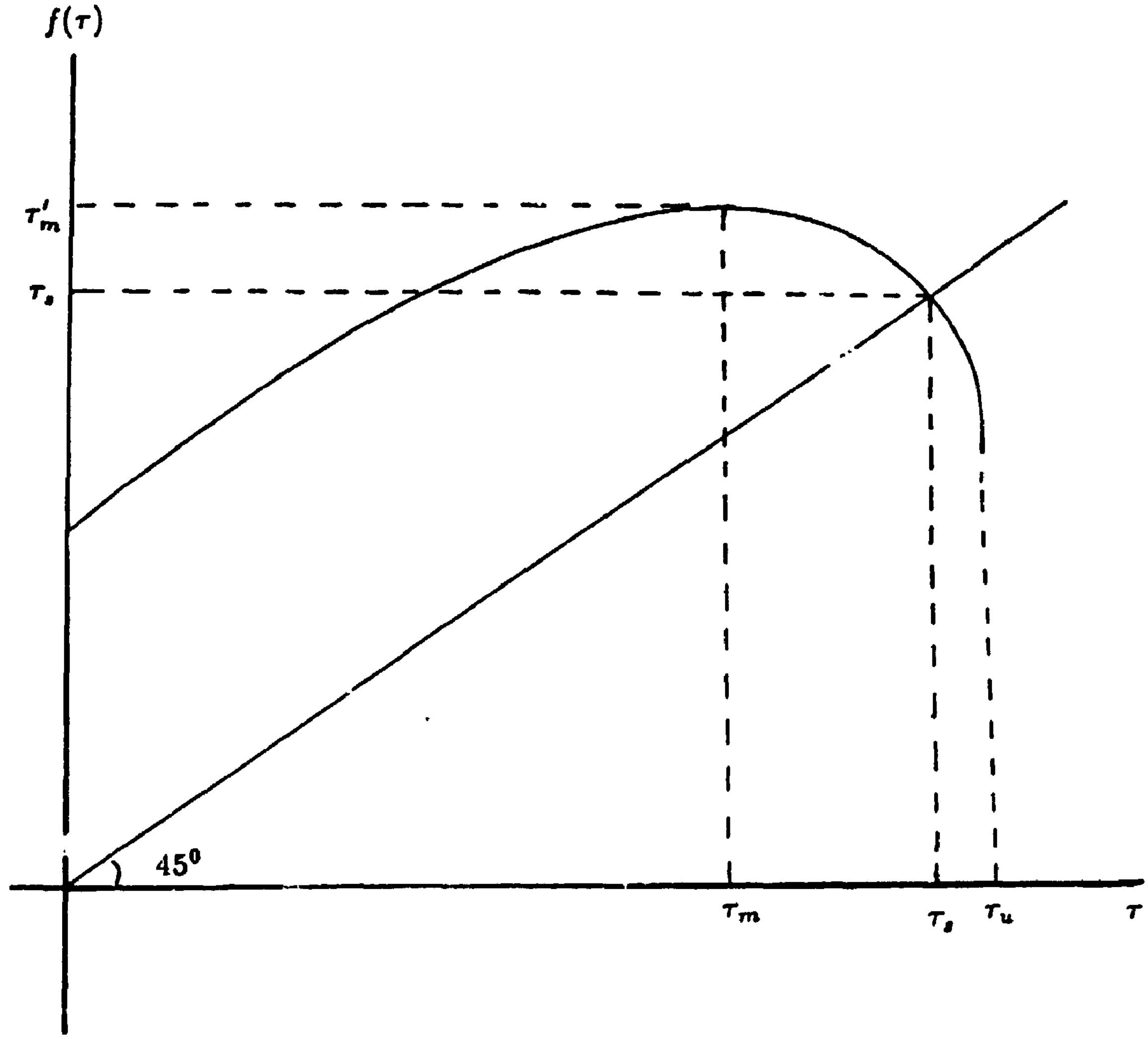
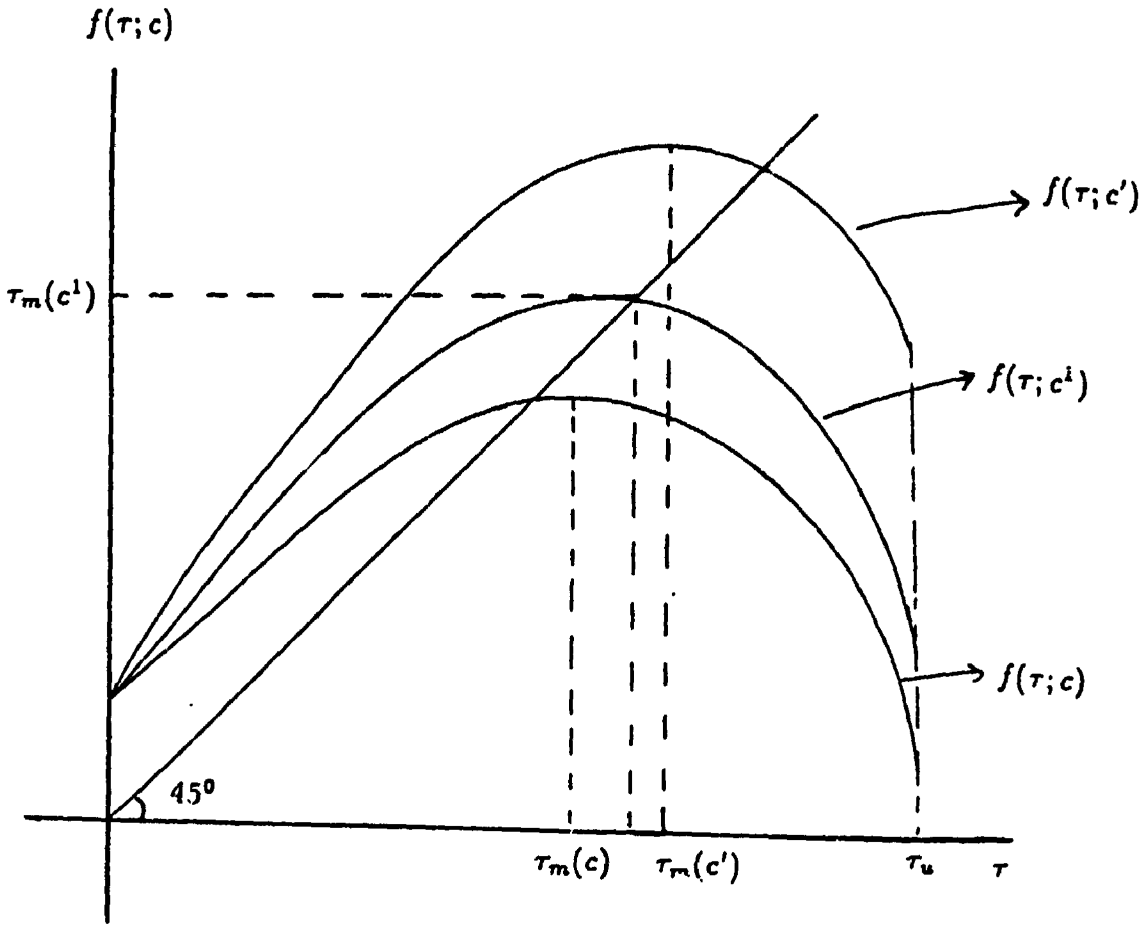


FIGURE 3.4.1.



Where: $c \in (c^1, 1]$ and $c' \in (c^0, c^1)$

FIGURE 3.4.2.

To summarize the above results, c^0 and c^1 ($c^0 < c^1$) divides the interval [0, 1] into three segments. For c that lies in the interval $[0,\ c^0]$, the value of $f(\tau_t; c)$ is greater than or equal to τ_u for some $\tau_t \in [0,\ \tau_u]$. In this case, a competitive equilibrium in the sense of Definition 3.3.1 does not necessarily exist. For c that lies in the interval $(c^0,\ 1]$, on the other hand, $f(\tau_t, c) < \tau_u$ for all $\tau_t \in [0,\ \tau_u]$ and hence a competitive equilibrium exists. Furthermore, the fixed point of $f(\tau_t; c)$, τ_s, locates at the upward-sloping portion of f for $c \in (c^1,\ 1]$ and at the downward-sloping portion of f for $c \in [c^0,\ c^1)$. (See Figure 3.4.2).

Remarks: In this model as the agriculture productivity falls, the equilibrium output of the manufactured good falls. As the agriculture productivity approaches 0, the equilibrium output of the manufactured good approaches 0 as well. What is going on in the case $c \in [0,\ c^0]$ is that, because of the assumed extremely small ability by the Earth to shed heat, the temperature rises at a much faster pace than the rise in relative price of the agriculture good in response to the greenhouse effect caused by manufacturing activities. As a result, starting from some τ_t, the global temperature may rise above τ_u before the market (prices) has time to react. In the real world, however, there is no reason to believe that the response of the market mechanism is any slower than the response of the climate system to the greenhouse effect. In fact, it appears that the contrary is true. Therefore, in the remainder of the chapter we shall concentrate on the case $c \in (c^0,\ 1]$.

Lemma 3.4.6. *Assume (A3.1)–(A3.7). There exists $c_f \in [c^0,\ c^1)$ such that $f(\tau_m'(c)) > \tau_m(c)$ for all c in the interval $(c_f,\ c^1)$.*

Proof: See Appendix II.1.

Notice that c_f is not unique. In what follows, c_f refers to the smallest of such c only. If $c_f = c^0$, $f(\tau_m') > \tau_m$ for all c in the interval $(c^0,\ c^1)$.

3.5. Equilibrium Time Path of Global Temperature: Case 1

Given the results obtained in Section 3.4, the behavior of the equilibrium time path of the global temperature can be determined. As shown in the last section c^1 divides the interval $(c^0, 1]$ into two sub-intervals. The time path of global temperature has different characteristics for c in the different sub-intervals. In this section, the case of c in the intervals $[c^1, 1]$ is analysed. The analysis of the case $c \in (c^0, c^1]$ is left to Section 3.6 and Section 3.7.

The analysis in the next three sections draws heavily on definitions and results on the theory of one-dimensional dynamical system as presented in Collet and Eckmann (1980) (CE) and Grandmont (1986). Appendix II.2 is a brief list of the related definitions and theorems drawn from these two sources. Several important definitions, however, are presented below in the text.

Let $f^i(\tau)$ denote the i^{th} iterations of a function $f(\tau)$, ie, $f^i = f \circ f^{i-1}$ for $i = 1$, 2, ..., and $f^0(\tau) \equiv \tau$.

Definition 3.5.1. *Consider the difference equation $\tau_{t+1} = f(\tau_t)$, with f being a continuous function that maps the interval $[0, \tau_u]$ into itself. The orbit of τ is the set $\{\tau, f(\tau), f^2(\tau) \dots\}$. τ is said to be a periodic point of f with period k if (1) τ is a fixed point of f^k, ie, $\tau = f^k(\tau)$; and (2) k is the smallest integer having this property, ie, $\tau \neq f^i(\tau)$ for all $i = 1, \dots, k-1$. The corresponding periodic orbit (or cycle) is the set $\{\tau_1, \dots, \tau_k\}$ of all the iterates $\tau_i = f^{i-1}(\tau)$ of τ, for $i = 1, \dots, k$.*

Definition 3.5.2. *A periodic orbit of f, $\{\tau_1, \dots, \tau_k\}$, is locally stable if $|Df^k(\tau_1)| < 1$. The periodic orbit is said to be weakly stable if $|Df^k(\tau_1)| \leq 1$. It is said to be superstable if $Df^k(\tau_1) = 0$.*

Note: if a periodic orbit is stable, there exists a neighbourhood U of τ_1 such that for every τ in U, $f^{kj}(\tau)$ stays in U for all $j \geq 1$ and $\lim_{j \to \infty} f^{kj}(\tau) = \tau_1$. Weak stability of a cycle allows for "one-sided" stability while superstability implies that the critical point of f, τ_m, belongs to the periodic orbit.

Let μ be Lebesgue measure.

Definition 3.5.3. *f displays sensitive dependence on initial conditions if there exists a set $Y \subset [0, \tau_u]$ with $\mu(Y) > 0$ and $\varepsilon > 0$ with the following property: for every $x \in Y$ and every neighbourhood U of x, there is $y \in U$ and integer $n \geq 0$ such that $|f^n(x) - f^n(y)| > \varepsilon$.*

The definition states that, if f has sensitive dependence on initial conditions, then no matter how close x and y are, they will eventually be noticeably separated under the repeated action of f. This means that the orbit of τ (under the repeated action of f) will depend in a sensitive way on the choice of the initial point τ_0.

Proposition 3.5.1. *Assume (A3.1)-(A3.7). If the value of c is in the interval $[c^1, 1]$, then the fixed point of f, τ_s, is the unique periodic orbit of f. Starting from $\tau_0 \in [0, \tau_u)$ the global temperature will monotonically approach τ_s which lies in the interval $[0, \tau_m]$.*

Proof: Obvious from Lemma 3.4.3.

Q.E.D.

Notice that $\tau_s < \bar{\tau}$ for c close to 1 and $\tau_s > \bar{\tau}$ for c close to c^1. Hence in this case, depending on the magnitude of c, we may or may not observe falls in the world agriculture productivity. But in any event, if c is in the interval $[c^1, 1]$, the price mechanism functions to stop the trend of rising temperatures and the world economy approaches a steady state.

3.6. Equilibrium Time Path of Global Temperature: Case 2

The case of $c \in (c^0,\ c^1)$ is more complicated than the one in Section 3.5. In this section equilibrium is studied under the assumption that $c_f = c^0$. Section 3.7 analyses the equilibrium time path when $c_f > c^0$.

When $c \in (c^0,\ c^1)$, $f(\tau) < \tau_u$ for all $\tau \in [0,\ \tau_u]$. Hence f is a unimodal that maps $[0,\ \tau'_m]$ into itself. Furthermore, it can be shown that for any $\tau_0 \in [0,\ \tau_u)$, there exists an integer N, such that $\forall n \geq N\ f^n(\tau_0) \in [f(\tau'_m),\ \tau'_m]$.

Proposition 3.6.1. *Assume (A3.1)-(A3.7). If $c \in (c^0,\ c^1)$ and $c_f = c^0$, then for any $\tau_0 \in [0,\ \tau_u)$, τ_t converges to either a periodic orbit of order 1 (ie, a steady state) or a periodic orbit of order 2 that lies in the interval $[f(\tau'_m),\ \tau'_m]$.*

Proof: See Appendix II.3.

Proposition 3.6.2. *Assume (A3.1)-(A3.7). If $c \in (c^0,\ c^1)$ and $c_f = c^0$, then $[f(\tau'_m)\ ,\tau'_m]$ contains a weakly stable periodic orbit of f.*

Proof: See Appendix II.3.

Proposition 3.6.1 states that for any intial temperature in the interval $[0,\ \tau_u)$, $\{\tau_t\}_{t=0}^{\infty}$ will converge to a periodic orbit which may or may not be stable. Proposition 3.6.2 states that at least for some $\tau_0 \in [0,\ \tau_u)$ the global temperature will converge to a stable periodic orbit.

Stronger results can be obtained by assuming negative Schwarzian derivative, ie,

(A3.8) f is C^3 and satisfies

$$S[f(\tau)] \equiv \frac{D_\tau^3 f}{D_\tau f} - \frac{3}{2}\left[\frac{D_\tau^2 f}{D_\tau f}\right]^2 < 0$$

whenever $D_\tau f \neq 0$.

Proposition 3.6.3. *Assume (A3.1)-(A3.8). If $c \in (c^0,\ c^1)$ and $c_f = c^0$, then there* *exists a unique weakly stable periodic orbit P which lies in the interval $[f(\tau_m'),\ \tau_m']$ and which attracts the critical point τ_m. The period of P is 1 or 2.*

Proof: Follows from Propositions 3.6.1 and 3.6.2 and Grandmont's (1986) Proposition 3.

Q.E.D.

Proposition 3.6.4. *Assume (A3.1)-(A3.8), $c \in (c^0,\ c^1)$, and $c_f = c^0$. There exist at most two periodic orbits, one of which is weakly stable. Furthermore, (a) if $-D_\tau g(\tau_s) \leq 2 - c$, then the fixed point, τ_s, is the unique periodic orbit; (b) if $-D_\tau g(\tau_s) > 2 - c$, then, in addition to the fixed point, there exists a (unique) two-period cycle P. P is weakly stable.*

Proof: See Appendix II.3.

Proposition 3.6.5. *Assume (A3.1)-(A3.8), $c \in (c^0,\ c^1)$, and $c_f = c^0$. Let E be the set of τ_0 such that $f^n(\tau)$ does not converge to P. Then $\mu(E) = 0$. In other words, $f^n(\tau)$ converges to P for almost all $\tau_0 \in [0,\ \tau_u)$.*

Proof: Follows from Proposition 3.6.3 and CE II.5.7.

Q.E.D.

Proposition 3.6.6. *Assume (A3.1)-(A3.8). If $c \in (c^0,\ c^1)$ and $c_f = c^0$, then f is not sensitive to initial conditions.*

Proof: Follows from Proposition 3.6.5 and CE II.7.1.

Q.E.D.

The results in this section have been obtained under the assumption $c_f = c^0$. The following proposition offers a necessary and sufficient condition under which $c_f = c^0$.

Proposition 3.6.7. *Assume (A3.1)-(A3.7). $c_f = c^0$ if and only if*

$$-\frac{g(\tau_m') - c\tau_m'}{g(\tau_m) - c\tau_m} < 1 \tag{3.6.1}$$

for all $c \in (c^0,\ c^1)$.

Proof: See Appendix II.3.

Recall that $g(\tau_t)$ represents the increase in next period's temperature caused by manufacturing activities while $c\tau_t$ is the reduction in next period's temperature by nature when the current temperature is at τ_t. Suppose that temperature reaches τ_m in period T. Then temperature will reach its maximum level, τ_m', in period $T+1$ and will fall in period $T+2$. (3.6.1) states that the magnitude of decrease in temperature between period $T+1$ and $T+2$ must be smaller than the magnitude of increase in temperature between T and $T+1$. Therefore, Proposition 3.6.7 implies that the fluctuation in the size of manufacturing activities must be relatively small in order to obtain simple dynamics (steady state or two-period cycle) for temperature.

Proposition 3.6.7 can be restated in terms of restrictions on the greenhouse effect term $g(\tau)$.

Corollary 3.6.8. *Assume (A3.1)-(A3.7).* $c_f = c^0$ *if and only if* $g(\tau)$ *satisfies*

$$-\frac{g[g(\tau)(1+\eta_{g\tau})] - [1 + D_\tau g(\tau)]g(\tau)(1+\eta_{g\tau})}{g(\tau) - (1 + D_\tau g(\tau))\tau} < 1 \qquad (3.6.2)$$

for all $\tau \in (\tau_m(c^1),\ \tau_m(c^0))$ *with* $\eta_{g\tau} \equiv -\frac{\tau D_\tau g(\tau)}{g(\tau)}$.

Proof: See Appendix II.3.

Remarks on Assumption (A3.8): The Schwarzian derivative plays an important role in the theory of one-dimensional dynamical systems. The implications of this condition in the context of this model, however, are not easy to derive since it involves third-order derivatives whose expressions are complicated. The remainder of this section attempts to relate the Schwarzian derivative to the primitives of the model by offering a set of necessary conditions for $S[f(\tau)] < 0$ and some examples where these conditions may (or may not) be satisfied.

Proposition 3.6.9. *Assume (A3.1)-(A3.6). At least one of the following must be true for* $S[f(\tau)] < 0$*: either (a)* $S[g(S_2)] < 0$*; or (b)* $S[p\xi(p)] < 0$*; or (c)* $S[a(\tau)] < 0$*;*

or (d) $D_\tau^3 f < 0$. Furthermore, conditions (a)–(d) holding simultaneously is sufficient for $S[f(\tau)] < 0$.

Proof: $S[f(\tau)] < 0$ if and only if

$$(D_\tau^3 f)(D_\tau f) - \frac{3}{2}(D_\tau^2 f)^2 < 0. \tag{3.6.3}$$

Calculation gives

$$\begin{aligned}
&(D_\tau^3 f)(D_\tau f) - \frac{3}{2}(D_\tau^2 f)^2 \\
&= [(D_{s_2}^3 g)(D_{s_2} g) - \frac{3}{2}(D_{s_2}^2 g)^2](D_p S_2)^4 (D_\tau p)^4 \\
&+ [D_p^3(p\xi(p)) D_p(p\xi(p)) - \frac{3}{2}(D_p^2(p\xi(p)))^2](D_{(p\xi(p))} S_2)^2 (D_{s_2} g)^2 (D_\tau p)^4 \\
&+ (\frac{1}{b})^2[(D_\tau^3 a)(D_\tau a) - \frac{3}{2}(D_\tau^2 a)^2](D_{s_2} g)^2 (D_p S_2)^2 \\
&+ (1 - c) D_\tau^3 f.
\end{aligned} \tag{3.6.4}$$

At least one of the four terms in the right-hand side of (3.6.4) must be negative for $S[f(\tau)] < 0$ to hold. Furthermore, $S[f(\tau)] < 0$ if all four terms are negative.

Q.E.D.

Examples can be found where specific functional forms of $g(S_2)$, $p\xi(p)$ and $a(\tau)$ have the property of negative Schwarzian derivative. It can be verified that $S[g(S_2)] < 0$ if $g(S_2)$ is a third-degree polynomial in S_2 with appropriately chosen parameter values. It is also easy to check that $a(\tau)$ has a negative Schwarzian derivative if $a(\tau)$ is a quadratic function in τ. The function $p\xi(p)$ derived from a CES utility function satisfies the condition of negative Schwarzian derivative if the elasticity of substitution is greater than 2. This last condition, however, contradicts Assumption (A3.3).

Consider an example where the functional forms of $g(S_2)$ and $a(\tau)$ are as described above and where the utility function is CES with the elasticity of substitution less than unity. It can be shown that $D_\tau^3 f < 0$ for $\tau > \bar{\tau}$ and $D_\tau^3 f > 0$ for $\tau < \bar{\tau}$. Since $g(S_2)$ and $a(\tau)$ have negative Schwarzian derivatives, $S[f(\tau)] < 0$ can be satisfied as long as the first and the third terms dominate the second and the forth terms in the right-hand side of (3.6.4).

3.7. The Emergence of Chaotic Climate

In this section, the possibility of the emergence of chaotic climate is explored. In Section 3.6, it has been shown that under (A3.1)-(A3.7) and $c_f = c^0$, the longest possible periodic orbits are of period two. If $c_f > c^0$, however, it is possible that competitive equilibrium leads to a climatic chaos, as we shall see in this section.

Define τ_k as the smaller of τ that satisfies $f(\tau) = \tau_m$. ($\tau_k < \tau_m$). A climatic chaos may occur if there exists $c_e \in (c^0, c_f)$ such that $f(\tau_m'(c_e)) < \tau_k(c_e)$. In the following analysis, it is assumed that

(A3.9) c_e exists.

It is straightforward to verify that under (A3.1)-(A3.7) and (A3.9), $f(\tau; c)$ with $c \in [c_e, c^1]$ constitutes a full family of C^1-unimodal maps. The following results are direct applications of the results in Grandmont (1986).

Proposition 3.7.1. *Assume (A3.1)-(A3.7) and (A3.9). Consider $c \in [c_e, c^1]$.*

(a) Given an arbitrary $k \geq 2$, the set of c for which the map $f(\cdot; c)$ has a superstable cycle of period k is closed and non-empty. Given such a c, there is an open interval around c such that $f(\cdot; c')$ has a stable cycle of period k for all c' in this interval.

(b) Let c_j^* be the largest value of the parameters c for which a superstable cycle of period 2^j obtains for $j \geq 1$. Then the sequence c_j^* decreases with j and converges to some value $c_\infty^* > c_e$ as j tends to $+\infty$. For each c in $(c_\infty^*, c^1]$, all cycles of the map $f(\cdot; c)$ have a period that is a power of 2 or are fixed points. The critical point $\tau_m(c)$ of $f(\cdot; c)$ is attracted to one of these.

(c) If superstable cycles of periods 2^j and $2^{j'}$ with $j' > j + 1$ occur respectively for the values c and c' in $(c_\infty^*, c^1]$, then a superstable cycle of period 2^i with $j' > i > j$

must appear for some value in the open interval determined by c and c'.

Proof: See the proof of Grandmont's (1986) Theorem 7.

Q.E.D.

Proposition 3.7.2. *Assume (A3.1)–(A3.9).*

(a) For any c in $[c^*_\infty,\ c^1]$*, the map* $f(\cdot;c)$ *has a (unique) weakly stable periodic orbit.*

(b) There is an uncountable set of values of c in $[c_e,\ c^*_\infty)$ *for which* $f(\cdot;c)$ *has no stable periodic orbit.*

Proof: See Theorem 9 in Grandmont (1986).

Q.E.D.

Proposition 3.7.3. *Assume (A3.1)–(A3.9).*

(a) For any c in $[c^*_\infty,\ c^1]$*, the map* $f(\cdot;c)$ *has no sensitivity to initial conditions*

(b) There is an uncountable set of values of c in $[c_e,\ c^*_\infty)$ *for which* $f(\cdot;c)$ *has sensitivity to initial conditions.*

Proof: (a) follows from Proposition 3.7.2(a) and C.E. II.7.1 (see Appendix II.2). (b) follows from Grandmont (1986) Remark 1.

Q.E.D.

Since a necessary condition for $f(\cdot;c)$ to display sensitivity is that $f(\cdot;c)$ has no stable periodic orbit, the set of c values defined in Proposition 3.7.2(b) contains the set of c values in Proposition 3.7.3(b).

Remarks: In this section, it is demonstrated that under (A3.1)–(A3.9) c^*_∞ divides the interval $(c^0,\ c^1]$ into two sub-intervals. If $c \in (c^*_\infty,\ c^1]$ the world economy and temperature will converge to a unique steady state or a unique cycle of period 2^j ($j = 1$, 2, or 3...) for almost all initial conditions $\tau_0 \in [0,\ \tau_u)$. However, chaos and sensitive dependence on initial conditions may be observed if the value of c falls in the interval $(c^0,\ c^*_\infty)$. In particular, the interval $(c_e,\ c^*_\infty)$ contains an uncountable set of such c values[5].

The possible existence of sensitivity in the time path of temperature casts doubts on the existing projection about future climate changes. For a dynamical system that exhibits sensitivity, a small error of measurement of the initial state may result in very large prediction errors for future dates, even if the forecaster knows very well the law of motion of the system. In the reality, of course, we cannot even claim that we know the system well.

It is worth pointing out that chaotic climate may occur even if the climate system itself is stable. As we can see from (3.2.1), in the absence of any manufacturing activities the climate system has a globally stable steady state τ_n. It is the human activities that generated the possibility of chaotic climate. Hence, knowing the climate system alone is not enough for projecting future climate.

Intuitively speaking, the climatic cycles and chaos are a consequence of the dynamic interaction between the stable climate system and the self-stabilizing economic system. To illustrate this, consider the extreme cases where only one of these two systems controls the law of motion of temperature. First, we consider the case $c = 1$ (no accumulation of the greenhouse gases). The equilibrium law of motion of temperature becomes $\tau_{t+1} = g(S_{2t}(\tau_t))$. Temperature in period t is completely determined by the size of the manufacturing activities in the previous period. This is the case where the law of motion of temperature is controlled by the market mechanism. In this case, temperature converges to a steady state (Proposition 3.5.1) because the market mechanism is self-stabilizing. Second, consider the case where the market system does not respond to the temperature changes. To be more specific, suppose in each period there is a fixed amount of the greenhouse gases emitted into atmosphere that will raise temperature by $\bar{g}$ units. The law of motion of temperature in this case is $\tau_{t+1} = (1 - c)\tau_t + \bar{g}$. In this case, temperature again converges to a steady state, $\frac{k}{c}$, because the climate system is stable. Therefore, it is the interaction between the two stable systems that leads to the possibility of climatic cycles or chaos.

In this model, the climate system has the tendency to move temperature towards its steady state level. The market mechanism, at the same time, also works to reduce the output of the manufactured good when temperature exceeds $\bar{\tau}$. The combined forces of the two systems, under certain conditions, increase the amplitude of fluctuation. As a result, cycles or even chaos occur. Hence, in this model the self-stabilizing market mechanism does not always serve the function of stabilizing temperature. In fact, under certain conditions, the market mechanism works to "sabotage" the functioning of the climate system and to destabilize temperature.

Finally, the following result provides a sufficient condition under which (A3.9) is true.

Proposition 3.7.4. *Assume (A3.1)-(A3.7) and $c_f > c^0$. (A3.9) holds if there exists $c \in (c^0,\ c_f)$ such that for all $v \in [f(\tau'_m),\ \tau_m]$,*

$$-D^2_\tau g(v) > \frac{2(\tau'_m - \tau_m)}{[f(\tau'_m) - \tau_m]^2} \tag{3.7.1}$$

Proof: See Appendix II.3.

3.8. Concluding Remarks

This chapter offered an alternative approach to the greenhouse effect modelling by incorporating endogenous economic activities into a simple climate system model. It was demonstrated that even though the climate system itself is stable and the market mechanism works to offset rising temperatures, global temperature will still display a cyclical or even chaotic time path under certain conditions. Therefore, the answer to the question posed in the title is: yes, under certain conditions economic activities will indeed lead to a climatic chaos.

One of the main contributions of this chapter is that it points out and proves the possibility that the interaction between a stable natural system and a self-stabilizing market mechanism can lead to cyclical or even chaotic behavior. A built-in self-stabilizing market mechanism will not always serve the function of stabilization. Under certain conditions, it may increase the amplitude of fluctuations and have the effect of de-stabilization, as demonstrated in this chapter. Therefore, by incorporating a self-stabilizing market mechanism this model yields a result that is opposite to Solow's (1973) conjecture that the market mechanism will have the effects of smoothing the time path of the world economy. While Solow may be absolutely correct in the context of the Club of Rome Report, one should not take Solow's arguments as universally applicable to all other environmental issues such as the global warming.

One implication of the model is that the informal approach proposed by Lave (1982) may not be sufficient to generate reliable predictions. In this model, if one follows Lave's proposed approach by letting an economist and a climatologist independently compute their "internally consistent scenarios" using models of their own disciplines and then interact with each other, what will be generated from this process is that both will arrive at the conclusion that the world will reach a steady state. The possibility of cycles and chaos will inadvertently be neglected in this process. Therefore, a formal, unified general equilibrium model is needed in order to generate reliable predictions. A research project on future climate change should involve a team of researchers who are from all related disciplines including economics, with each member of the team being responsible for one part of the project that is related to his area of expertise.

In this model, it has been assumed that there is no capital good. The rationale for this assumption is that climate change is a very slow process. The atmospheric residence time of CO_2 is hundreds of years. Climate change is measured in terms of time scales that are in the magnitude of hundreds of years as well. Most capital goods depreciate completely within such a time period. Therefore, the issue of investment and capital accumulation can be considered irrelevant in a model with such a long period(6).

In this chapter, no attempt is made to calibrate the model. The main goals of this chapter are to generate qualitative predictions about the time path of temperature and at the same time to offer an example to illustrate the proposition that the dynamic interaction between a natural system and a self-stabilizing market mechanism can lead to cyclical or even chaotic behavior. The theoretical model presented in this chapter seems to be sufficient to accomplish these objectives. Nevertheless, the main propositions of the chapter would be more forceful if one could calibrate the model using parameter values that are within the range established by empirical studies and derive cyclical and chaotic behavior from the calibrated model. Thus calibration of the model is an interesting exercise that warrants future work.

Finally, it should be emphasized that while climatic cycles and chaos are possible, they are by no means inevitable. First, climatic cycles and chaos will occur only under certain conditions. It was shown in the chapter that the world will converge to a steady state under other conditions. Second, and more importantly, the results in this chapter are obtained under the implicit assumption of stationary social and economic structure. While it is extremely difficult to predict with certainty the changes in social and economic structure in the next one hundred years, it is conceivable that the world will undergo tremendous social and economic changes as the decades go by. (To see this possibility, one needs only to think of how much the world has changed in the past one hundred years.) These social and economic changes will undoubtedly have profound influences on the way in which the human race will interact with the nature and the way in which the climate changes will affect the well-being of the human race. (For a more detailed discussion on these issues, see Schelling, 1983).

Footnotes

(1) Most of the discussions and research conducted by economists are centered on the policy issues and the costs and benefits associated with combating the global warming. (See, for example, Whalley and Wigle, 1991; Manne and Richels, 1990; Nordhaus, 1990a,1990b).

(2) D. Brunt points out that the recent warming is no more striking than that of earlier periods during which human contributions to atmospheric CO_2 were negligible (Solow, 1989). Kenneth E.F. Watt, professor of environmental studies at the University of California at Davis, called the greenhouse effect "the laugh of the century." (*Time*, January 2, 1989, p22).

(3) The original model in Dickinson (1986) is in the form of a differential equation which can be written as (in our notions): $\frac{\partial \Delta\tau_t}{\partial t} + \frac{c}{\Delta t}\Delta\tau_t = g(S_{2t})$ with $\Delta\tau_t \equiv \tau_t - \tau_n$ and Δt being the length of a period. c is equal to Δt times the ratio of temperature feedback parameter to the system heat capacity parameter.

(4) While deforestation in order to expand agriculture activities itself also generates additional CO_2, the net emission from this source is insufficient to bring about a significant change of climate (Bolin Jäger and Döös, 1986, p7). Hence we assume here that the agricultural sector does not generate the greenhouse effect.

(5) Nothing is said about the case $c \in (c^0,\ c_e)$. This is because the analysis on $c \in [c_e,\ c^1]$ has generated all possible scenarios. An analysis on the case $c \in (c^0,\ c_e)$ can be conducted in the same way but will not generate any new scenarios.

(6) Nordhaus (1990a) discusses the difficulties in choosing appropriate time discount factors for computing the cost of future climate change. A rate close to the return on capital in most countries, Nordhaus explains, would imply that one should forget about climate change for a few decades. Hence a much lower time discount rate is required in order to derive plausible estimates of

future climate damages. As a result, one is faced with the dilemma of a low time discount rate on climate change and a high return on capital (Nordhaus, 1990a p205). One explanation for such a dilemma is that the rate of return on capital is irrelevant to the calculation of the future climate damages because capital goods are "perishable" when considered in terms of the time scale of the greenhouse effect models.

CHAPTER 4

Negotiation and Optimality in a Model of Global Warming

4.1. Introduction

In Chapter 3, it has been demonstrated that the time path of the world economy and climate may be cyclical or even chaotic under competitive equilibrium. There are two sets of questions that are left unanswered in Chapter 3: (1) What are the characteristics of the optimal time path of the world economy and climate? In particular, will the optimal time path converge to a steady state? (2) What will happen if the governments decide to correct the problem of global warming? Specifically, will they be able to achieve global optimality in competitive economies?

The purpose of this chapter is to answer these questions through a formal economic analysis. Using a two-country, two-sector general equilibrium model, the chapter shows that the global optimal time path of outputs and temperature will converge to a unique steady state provided that consumers care enough about the future. To answer the second set of questions, the chapter studies the equilibrium outcome in a bargaining game where two countries negotiate an agreement on future consumption and production plans for the purpose of correcting the problem of global warming. It is demonstrated that the agreement that arises from such a negotiation process achieves global optimality. It is also shown that the agreement can be implemented in decentralized economies by a system of taxes and transfers.

While most of the discussion in the literature about global environmental problems by economists points out the importance of international cooperation in coping with these problems (See, for example, Barrett 1990, Nordhaus 1990a), there are few formal economic modelling of these issues in the literature. The only exception is Markusen (1975) which studies the cooperative control of international pollution. However, the way in which countries cooperate with each other is exogenously assumed in Markusen (1975). In this chapter, by employing the recent advances in the non-cooperative bargaining theory, the agreement between the two countries is derived endogenously through a well specified bargaining procedure.

The remainder of the chapter is organized as follows. Section 4.2 describes the economic and natural environment of the model. Section 4.3 studies the characteristics of the global optimal time path of temperature and outputs by solving a world planner's problem. Sections 4.4 and 4.5 demonstrate how the global optimality can be achieved through an agreement negotiated by the two governments and the characteristics of such an agreement. It is shown in Section 4.6 that such an agreement can be implemented in decentralized economies by means of a system of taxes and international transfers. Section 4.7 is conclusions.

4.2. Economic and Natural Environment

In this section, the production, consumption and climatic aspects of the model are specified. Roughly speaking, we shall consider a world that consists of two countries. The two countries have the same preferences, the same production technology, the same climate, but (maybe) different population sizes. Two goods can be produced, one of which is an agricultural good and the other a manufactured good. The productivity of the agricultural sector is affected by the global temperature. The manufacturing activities, on the other hand, affect temperature level.

The formal specification of the model follows.

Time, denoted t, is discrete and the horizon is infinite: $t \in \{0, 1, \ldots\}$.

The world consists of two countries: Country H and Country F. Population in each country is constant over time. Let the size of the world population be normalized to one. Country H has a population of size α while Country F $1-\alpha$. Population is immobile between countries. The two countries are assumed to have identical production technology, identical preferences and identical climate. In what follows, the production and consumption side of the model is specified for Country H. The variables of Country F, which will be denoted by attaching a superscript "*", can be specified in the same way.

On the *production side* of the world, two non-storeable goods are produced: an agricultural good and a manufactured good, with quantities being denoted by S_1 and S_2 for H. Goods can be transported at zero cost.

There is a fixed continuum of firms in each industry in each country. Hence both industries are perfectly competitive. Labor is the only input of production. At each date a representative firm in industry i ($i = 1, 2$) in Country H chooses the level of employment in the industry, ϕ_{it}.

The production technology of both goods exhibits constant return to scale. The productivity of labor in the manufacturing sector does not depend on climate and is denoted by b. Country H's output of the manufacturing sector at date t can then be written as $S_{2t} = b\phi_{2t}$. In the agricultural sector, however, the productivity of labor depends on one aspect of the climate, namely the global temperature. Let $a(\tau_t)$ denote the productivity coefficient of the agricultural sector, ie, $S_{1t} = a(\tau_t)\phi_{1t}$, where τ_t is the world average temperature level in period t. It is assumed that $a(\tau_n) > 0$; $a'(\tau_t) > 0$ if $\tau_t < \bar{\tau}$ and $a'(\tau_t) < 0$ if $\tau_t > \bar{\tau}$; $a(\tau_u) = 0$; and $a''(\tau_t) < 0$. τ_n denotes the "natural" temperature level, ie, the level at which the global temperature would stay in the absence of any manufacturing activities. $\bar{\tau}$ is some critical level of temperature for the agriculture sector. $\tau_n < \bar{\tau}$. $\tau_u (> \bar{\tau})$ is the temperature level at which the agriculture productivity equals zero. Therefore, by assumption the agriculture productivity is positive when there have been no manufacturing activities. A higher level of temperature improves the agriculture productivity as long as the temperature is below the critical value $\bar{\tau}$. As temperature level exceeds $\bar{\tau}$, however, higher temperature will reduce the productivity of the agriculture sector. The agriculture productivity eventually approaches zero as temperature level reaches τ_u.

The *consumer side* of the economy comprises a fixed continuum of identical consumers. A representative consumer is endowed with one unit of labor endowment which is supplied inelastically. He has no initial wealth. His preference over consumptions of the agricultural good and the manufactured good at date t, denoted by C_{1t} and C_{2t}, respectively, is represented by the the utility function $\sum_{t=0}^{\infty} \beta^t U(C_{1t}, C_{2t})$, where $U(\cdot)$ satisfies:

(A4.1) $U(C_1, C_2) \in C^2$ satisfies $\lim_{C_1 \to 0} U_1(C_1, C_2) = +\infty$, $\lim_{C_2 \to 0} U_2(C_1, C_2) = +\infty$ and that $U(0,0) = 0$,

(A4.2) $U(C_1, C_2)$ is concave and homogenous of degree γ.

(A4.3) $U_{12}(C_1, C_2) \geq 0$.

The law of motion of the *global average temperature* is characterized by a variation of the zero-dimensional climate system model presented in Dickinson (1986):

$$\tau_{t+1} = (1 - c)(\tau_t - \tau_n) + \tau_n + g(S_{2t} + S_{2t}^*) \qquad (4.2.1)$$

where $c \in (0\ 1)$, $g(\cdot) \in C^2$, $g(0) = 0$, $g'(\cdot) > 0$ and $g''(\cdot) \leq 0$.

(4.2.1) states that the manufacturing activities raises temperature. When temperature is above its natural level, τ_n, the nature has the ability of absorbing a percentage of the excess greenhouse gases and cooling down the climate towards its natural level at a rate c. Since it is assumed that $\tau_n < \bar{\tau}$, starting from the point of time where no manufacturing activities had taken place in the past "some" manufacturing activities would be good for the production of the agriculture good by assumption.

Without any loss of generality, we choose the unit of temperature level in such a way that $\tau_n = 0$.

4.3. The Global Optimal Solution

In this section, the problem faced by a hypothetical "world central planner" is presented and solved. It is demonstrated that the sequence of optimal outputs converges to a unique steady state as long as consumers care enough about the future. The world welfare possibility frontier is derived.

The objective of the world central planner is to maximize the weighted sum of per capita utilities of the two countries over an infinite time horizon. Let $\psi(< 1)$ denote the weight attached to H's per capita welfare and $1 - \psi$ the weight attached to F's per capita welfare. Let L_{1t}, L_{2t} be the labor input allocated to sector i $(i = 1, 2)$. The world planner's problem can then be written as:

$$\max_{\{C_{1t}, C_{2t}, C^*_{1t}, C^*_{2t}, L_{1t}, L_{2t}, \tau_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\psi U(C_{1t}, C_{2t}) + (1 - \psi) U(C^*_{1t}, C^*_{2t})] \quad (4.3.1)$$

subject to

$$\alpha C_{1t} - (1 - \alpha) C^*_{1t} = a(\tau_t) L_{1t} = S_{1t} + S^*_{1t} \quad (4.3.2)$$

$$\alpha C_{2t} + (1 - \alpha) C^*_{2t} = b L_{2t} = S_{2t} + S^*_{2t} \quad (4.3.3)$$

$$L_{1t} + L_{2t} = 1 \quad (4.3.4)$$

$$\tau_{t+1} = (1 - c)\tau_t + g(S_{2t} + S^*_{2t}) \quad (4.3.5)$$

and $\tau_0 \in [0, \tau_u)$ given.

In the above problem, given the sequence of the world outputs $\{S_{1t} + S^*_{1t}, S_{2t} + S^*_{2t}\}_{t=0}^{\infty}$, the allocation of the consumption goods among the two countries is a static problem and is governed by the following standard first-order conditions:

$$\psi U_1(C_{1t}, C_{2t}) = (1 - \psi) \frac{\alpha}{1 - \alpha} U_1(C^*_{1t}, C^*_{2t}) \quad (4.3.6)$$

$$\psi U_2(C_{1t}, C_{2t}) = (1 - \psi) \frac{\alpha}{1 - \alpha} U_2(C^*_{1t}, C^*_{2t}) \quad (4.3.7)$$

Define $\omega = \frac{1-\psi}{\psi}\frac{\alpha}{1-\alpha}$. Notice that $\omega = 1$ if $\psi = \alpha$. (4.3.6)-(4.3.7) together with the assumption of homogeneity on $U(\cdot)$ implies

$$C_{1t} = \frac{S_{1t} + S_{1t}^{*}}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}} \tag{4.3.8}$$

$$C_{1t}^{*} = \frac{\omega^{\frac{1}{1-\gamma}}(S_{1t} + S_{1t}^{*})}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}} \tag{4.3.9}$$

$$C_{2t} = \frac{S_{2t} + S_{2t}^{*}}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}} \tag{4.3.10}$$

$$C_{2t}^{*} = \frac{\omega^{\frac{1}{1-\gamma}}(S_{2t} + S_{2t}^{*})}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}} \tag{4.3.11}$$

Since $g(\cdot)$ is monotonically increasing in its argument, one can define its inverse function $G(\cdot) \equiv g^{-1}(\cdot)$. (4.3.5) can be written as

$$S_{2t} + S_{2t}^{*} = G(\tau_{t+1} - (1-c)\tau_t) \tag{4.3.12}$$

The assumptions on $g(\cdot)$ implies that $G(0) = 0$, $G'(\cdot) > 0$ and $G''(\cdot) \geq 0$.

Using (4.3.2)-(4.3.4), (4.3.8)-(4.3.12) and the homogeneity property of $U(\cdot)$, one can rewrite (4.3.1) as

$$W(\tau_0;\psi) = \max_{\{\tau_{t+1}\}_{t=0}^{\infty}} [\psi(\frac{1}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}})^{\gamma} + (1-\psi)(\frac{\omega^{\frac{1}{1-\gamma}}}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}})^{\gamma}] \cdot$$
$$\cdot \sum_{t=0}^{\infty} \beta^t U\left[a(\tau_t)\left(1 - \frac{1}{b}G(\tau_{t+1} - (1-c)\tau_t)\right), G(\tau_{t+1} - (1-c)\tau_t)\right] \tag{4.3.13}$$

Notice that if $\psi = \alpha$, $W(\tau_0;\psi)$ is independent of ψ and α. Therefore, we can define $W(\tau) \equiv W(\tau;\alpha)$.

Proposition 4.3.1. *Assume (A4.1)-(A4.2). The global optimal sequence of outputs* $\{S_{1t} + S_{1t}^{*}, S_{2t} + S_{2t}^{*}\}_{t=0}^{\infty}$ *is independent of* ψ *and* $1-\psi$.

Proof: The solution to (4.3.13) is the same as the solution to

$$W(\tau_0) = \max_{\{\tau_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U\left[a(\tau_t)\left(1 - \frac{1}{b}G(\tau_{t+1} - (1-c)\tau_t)\right), G(\tau_{t+1} - (1-c)\tau_t)\right] \tag{4.3.14}$$

which is independent of ψ.

Q.E.D.

The assumption of homogeneity of $U(\cdot)$ is crucial in obtaining Proposition 4.3.1. The homogeneity of $U(\cdot)$ implies that in any period the two countries will consume the two goods at the same proportion. When the weight parameter ψ changes, what the world planner has to do is reducing the consumption of both goods in one country and increasing the consumption of both goods in the other country by the same proportion, leaving the optimal world output mix unchanged.

Proposition 4.3.1 suggests that the sequence of optimal temperature can be solved from (4.3.14). Define

$$V(\tau_t, \tau_{t+1}) \equiv U\left[a(\tau_t)(1 - \frac{1}{b}G(\tau_{t+1} - (1-c)\tau_t)), G(\tau_{t+1} - (1-c)\tau_t)\right] \quad (4.3.15)$$

In order to apply the technique of dynamic programming, we define two sets. Let T denote the set of feasible values for the state variable τ_t. $T = [0,\ \tau_u]$ by definition. Let $\Gamma(\tau)$ be the set of feasible values for the state variable next period given the current state $\tau \in T$. (4.3.3)–(4.3.5) implies that $\Gamma(\tau) = [(1-c)\tau,\ \min\{(1-c)\tau + g(b),\ \tau_u\}]$. Let A be the graph of Γ:

$$A = \{(\tau, \tau') \in T \times T : \tau' \in \Gamma(\tau)\}. \quad (4.3.16)$$

The following assumptions are needed in establishing the concavity of $V(\cdot)$:

(A4.4) $-\frac{g''(S)S}{g'(S)} < 1 - \gamma$.

(A4.5) $-\frac{a(\tau)a''(\tau)}{(a'(\tau))^2} > \frac{1}{1-\gamma}$.

Lemma 4.3.2. *T is a convex subset of $\mathcal{R}$, and the correspondence $\Gamma : T \to T$ is nonempty, compact-valued and continuous. Furthermore, Γ is convex in the sense that, for any $\theta \in (0,\ 1)$ and $\tau_a,\ \tau_b \in T$, $\tau'_a \in \Gamma(\tau_a)$ and $\tau'_b \in \Gamma(\tau_b)$ implies*

$$\theta\tau'_a + (1-\theta)\tau'_b \in \Gamma[\theta\tau_a + (1-\theta)\tau_b].$$

Proof: Obvious from the definition of T and Γ.

Q.E.D.

Lemma 4.3.3. *Assume (A4.1)-(A4.5). The function $V : A \to \mathcal{R}$ is continuous, bounded and strictly concave. Furthermore, V is continuously differentiable in the interior of A.*

Proof: The continuity and differentiability of V is obvious given (A4.1) and the specifications of $g(\cdot)$ and $a(\cdot)$. $V(\cdot)$ is bounded below by 0 and bounded above by $U(a(\bar{\tau}), b)$.

$$
\begin{aligned}
V_{22} = & U_{11}(\cdot)(\frac{a}{b})^2[G'(\cdot)]^2 - 2U_{12}(\cdot)\frac{a}{b}[G'(\cdot)]^2 - \frac{1}{1-\gamma}U_{12}(\cdot)C_1G''(\cdot) \\
& - U_1(\cdot)\frac{a}{b}G''(\cdot) - U_{22}(\cdot)C_2G''(\cdot)[-\frac{g'(S)}{Sg''(S)} - \frac{1}{1-\gamma}] < 0
\end{aligned}
\tag{4.3.17}
$$

It can be shown that $V_{11}V_{22} - V_{12}^2 > 0$ given (A4.1)-(A4.5). Therefore, $V(\tau_t, \tau_{t+1})$ is concave in (τ_t, τ_{t-1}).

Q.E.D.

Consider the functional equation

$$
W(\tau) = \max_{\tau' \in \Gamma(\tau)} [V(\tau, \tau') + \beta W(\tau')]. \tag{4.3.18}
$$

Proposition 4.3.4. *Assume (A4.1)-(A4.5). There exists a continuous, single-valued function $h_\beta(\tau)$ that solves the functional equation (4.3.18). Furthermore, W is bounded and strictly concave.*

Proof: Lemmas 4.3.2 and 4.3.3 establish conditions needed to apply Theorems 4.6 and 4.8 in Stokey and Lucas (1989).

Q.E.D.

Proposition 4.3.5. *Assume (A4.1)-(A4.5). $\tau_{t+1} = h_\beta(\tau_t)$ solves (4.3.14).*

Proof: Since $W(\tau)$ is bounded, $\lim_{n\to\infty} \beta^n W(\tau) = 0$. The result follows from Theorem 4.3 of Stokey and Lucas (1989).

Q.E.D.

The first-order condition to (4.3.18) can be written as:

$$-V_2(\tau_t, \tau_{t+1}) = \beta W'(\tau_{t+1}) = \beta V_1(\tau_{t+1}, \tau_{t+2}) \tag{4.3.19}$$

Propositions 4.3.4 and 4.3.5 state that the solution to (4.3.19), $\tau_{t+1} = h_\beta(\tau_t)$, is continuously differentiable and that $W(\tau)$ is concave.

Lemma 4.3.6. *Assume (A4.1)-(A4.5). There exists $\tau_s \in (0, \tau_u)$ such that $\tau_s = h_\beta(\tau_s)$.*

Proof: Substitute τ_s for τ_{t+i} $(i = 0, 1, 2)$ in (4.3.19):

$$-V_2(\tau_s, \tau_s) = \beta V_1(\tau_s, \tau_s) \tag{4.3.20}$$

Using (4.3.15) one can rewrite (4.3.20) as:

$$\begin{aligned}&[1 - \beta(1 - c)]G'(c\tau_s)\left[\frac{a(\tau_s)}{b} - \frac{U_2\big(a(\tau_s)(1 - b^{-1}G(c\tau_s)), G(c\tau_s)\big)}{U_1\big(a(\tau_s)(1 - b^{-1}G(c\tau_s)), G(c\tau_s)\big)}\right] \\ &= \beta a'(\tau_s)(1 - \frac{1}{b}G(c\tau_s))\end{aligned} \tag{4.3.21}$$

Denote the left-hand-side of (4.3.21) by LHS and the right-hand-side RHS. Since $a(0) > 0$ and $G(0) = 0$, then $S_1 + S_1^* > 0$ and $S_2 + S_2^* = 0$ if $\tau_s = 0$, which implies that $\lim_{\tau \to 0} \frac{U_2(\cdot)}{U_1(\cdot)} = +\infty$. Notice both $G'(\cdot)$ and $a'(\cdot)$ are finite. Therefore, LHS$< 0 <$RHS as τ_s approaches 0.

$a(\tau_u) = 0$ implies that $S_1 + S_1^* = 0$ and that $S_2 + S_2^* > 0$ if $\tau_s = \tau_u$. Thus $\lim_{\tau \to \tau_u} \frac{U_2(\cdot)}{U_1(\cdot)} = 0$. Hence, LHS$> 0 >$RHS as τ_s approaches τ_u.

By continuity there exists $\tau_s \in (0, \tau_u)$ such that (4.3.21) holds.

Q.E.D.

Lemma 4.3.7. *Assume (A4.1)-(A4.5). There exists $\bar{\beta} < 1$ so that if $\beta > \bar{\beta}$, τ_s is unique.*

Proof: The strict concavity of V implies that there exists a $\bar{\beta} < 1$ so that $\beta V_{11} + V_{22} + (1 + \beta)V_{12} < 0$ for all $\beta > \bar{\beta}$. Therefore, for all $\beta \in (\bar{\beta}, 1)$, $V_2(\tau, \tau) + \beta V_1(\tau, \tau)$ is a decreasing function in τ, which implies that τ_s is unique.

Q.E.D.

Lemma 4.3.8. *Assume (A4.1)-(A4.5). There exists $\hat{\beta} < 1$ so that if $\beta > \hat{\beta}$, $h_\beta(\tau)$ has no periodic point of period $n \geq 2$.*

Proof: This result is essentially the same as Lemma 3 in Deneckere and Pelikan (1986). Since $V(\tau, \tau')$ is strictly concave and $h_\beta(\tau)$ is continuous, Deneckere and Pelikan's (1986) proof can be directly applied with proper changes in notations.

Q.E.D.

Proposition 4.3.9. *Assume (A4.1)-(A4.5). There exists $\bar{\beta} < 1$ so that if $\beta > \bar{\beta}$, $\{h_\beta^j(\tau)\}_{j=1}^{\infty}$ converges to a unique steady state, τ_s, for all $\tau_0 \in [0, \tau_u)$.*

Proof: $h_\beta(\tau)$ is a continuous function that maps $[0, \tau_u]$ into itself. Define $\bar{\beta} = \max(\bar{\beta}, \hat{\beta})$. The result follows Lemmas 4.3.7 and 4.3.8 and the main theorem in Coppel (1955).

Q.E.D.

Proposition 4.3.9 states that the global optimal temperature will converge to a unique steady state in the long-run provided that consumers care enough about the future.

Given the sequence of optimal temperatures, the sequence of optimal outputs can be determined accordingly using (4.3.2)-(4.3.4) and (4.3.12). Next we shall study how the optimal outputs are allocated among the two countries. Specifically, we shall find out how the per capita welfare of each country varies with ψ. Define $\pi \equiv \sum_{t=0}^{\infty} \beta^t U(C_{1t}, C_{2t})$ and $\pi^* \equiv \sum_{t=0}^{\infty} \beta^t U(C_{1t}^*, C_{2t}^*)$. π and π^* are discounted sum of the per capita utility for H and F, respectively. Using (4.3.8)-(4.3.11) and homogeneity of $U(\cdot)$, one obtains:

$$\pi = \left(\frac{1}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}}\right)^\gamma W(\tau_0) \tag{4.3.22}$$

$$\pi^* = \left(\frac{\omega^{\frac{1}{1-\gamma}}}{\alpha + (1-\alpha)\omega^{\frac{1}{1-\gamma}}}\right)^\gamma W(\tau_0) \tag{4.3.23}$$

Given τ_0, $W(\tau_0)$ is a constant. The value of π and π^* depends on ψ.

Proposition 4.3.10. *Assume (A4.1)–(A4.5).* $\pi = \pi^* = W(\tau_0)$ *if* $\psi = \alpha$. $\pi > \pi^*$ *if* $\psi > \alpha$. $\pi < \pi^*$ *if* $\psi < \alpha$.

Proof: Obvious from (4.3.22) and (4.3.23)

Q.E.D.

If the world central planner assigns the welfare weight in proportion to the country's population size, every citizen in the world will enjoy the same level of welfare. If a country is assigned a weight that is larger than her proportion in the world population, however, a citizen in this country will have a higher welfare level than in the other country.

It is easy to verify that as ψ increases, π increases but π^* decreases. Therefore, we can define "the world welfare possibility frontier" by $\pi^* = v^*(\pi)$ where $v^*(\pi)$ is solved from (4.3.22) and (4.3.23):

$$\pi^* = v^*(\pi) = (1-\alpha)^{-\gamma}(W^{\frac{1}{\gamma}}(\tau_0) - \alpha\pi^{\frac{1}{\gamma}})^{\gamma} \qquad (4.3.24)$$

It is clear that π' is decreasing in π. Notice that if $\gamma = 1$, (4.3.24) is a linear function that can be written as

$$\alpha\pi + (1-\alpha)\pi^* = W(\tau_0) \qquad (4.3.25)$$

4.4. Negotiation

In the next two sections, the outcome of a negotiation process between the two countries are studied. In these two sections, the assumption of competitive economies is abandoned temporarily and is replaced by the assumption that there is a central planner in each country. The two central planners can negotiate a binding agreement on future production and consumption plans. The equilibrium outcome of this negotiation process is derived in Section 4.4 while the properties of the equilibrium outcome is analyzed in Section 4.5.

Consider a world where competitive equilibrium has been prevailing in the past. To simplify analysis, assume that the world has reached a steady state under competitive equilibrium(1). Let τ_c denote the steady state temperature under competitive equilibrium. $\tau_0 = \tau_c$ by assumption. Assume that in each country, there is a central planner who has the authority to choose consumption and production plans for the citizens in his country. The objective function of the central planner is to maximize the per capita utility of his citizens over an infinite time horizon. The central planners are identified by the country he represents (ie, H or F).

Suppose that in period 0, the two planners suddenly realize the importance and urgency of the global warming problem and decide to negotiate an agreement on future production and consumption plans for the two countries in an effort to correct this problem. The negotiation procedure is similar to that of the Rubinstein bargaining game. At the beginning of period 0, H proposes a sequence of consumption and production plans over an infinite time horizon for both countries. F immediately replies "Yes" or "No". If F says "Yes", an agreement is reached and the two planners will implement the agreement from period 0 on. If F replies "No", the planners will continue to allow the competitive equilibrium to prevail in period 0 and wait till period 1 when the second round of negotiation begins. At the beginning of period 1, the same process is repeated except that now F makes a proposal to which H immediately replies, and so on. We shall seek the subgame perfect equilibrium of this game.

Define $\pi(s) \equiv \sum_{t=s}^{\infty} \beta^t U(C_{1t}, C_{2t})$ and $\pi^*(s) \equiv \sum_{t=s}^{\infty} \beta^t U(C_{1t}^*, C_{2t}^*)$. $\pi(s)$ and $\pi^*(s)$ are the payoffs received by the two countries in a subgame starting from period s (s = 0, 1, 2,...). In any subgame starting from period s (s = 0, 2, 4,...), the problem faced by H is:

$$\max \pi(s) \tag{4.4.1}$$

subject to (4.3.2)-(4.3.5) and

$$\pi^*(s) \geq \bar{\pi}^* \tag{4.4.2}$$

where $\bar{\pi}^*$ is the minimum payoff that H believes F must obtain in a subgame perfect equilibrium. The problem faced by F in any subgame starting from period s (s = 1, 3, ...) is symmetric to the problem (4.4.1)-(4.4.2) faced by H.

Proposition 4.4.1. *Assume (A4.1)-(A4.5). Any subgame perfect equilibrium in this game is globally optimal.*

Proof: It can be verified that the problem (4.4.1)-(4.4.2) generates the same set of first-order conditions as the world planner's problem (4.3.1) with appropriately chosen ψ. Therefore, the proposal made by H or F in any subgame is globally optimal.

Q.E.D.

Therefore, any subgame perfect equilibrium must be a point on the world welfare possibility frontier as given by (4.3.24). In other words, any subgame perfect equilibrium will generate the sequence of optimal world outputs, denoted by $\{\hat{S}_{1t} + \hat{S}_{1t}^*, \hat{S}_{2t} + \hat{S}_{2t}^*\}_{t=0}^{\infty}$.

Let $\hat{\pi}$ and $\hat{\pi}^*$ be the subgame perfect equilibrium payoff to H and F, respectively. Suppose the sequence $\{\hat{C}_{1t}, \hat{C}_{2t}, \hat{C}_{1t}^*, \hat{C}_{2t}^*\}_{t=0}^{\infty}$ yields a payoff $\hat{\pi}$ to H and $\hat{\pi}^*$ to

F. (4.3.8)-(4.3.11) and Proposition 4.4.1 imply that $\hat{C}_{it}$, $\hat{C}^*_{it}$ $(i = 1, 2)$ are proportional to the world output of good i. Define λ as Country H's share of consumption of good i in the world total output of good i, ie,

$$\alpha \hat{C}_{it} = \lambda(\hat{S}_{it} + \hat{S}^*_{it}) \tag{4.4.3}$$

Then,

$$(1-\alpha)\hat{C}^*_{it} = (1-\lambda)(\hat{S}_{it} + \hat{S}^*_{it}) \tag{4.4.4}$$

Using (4.4.3)-(4.4.4) and the homogeneity assumption of utility function, one obtains:

$$\frac{\hat{\pi}}{\hat{\pi}^*} \equiv \frac{\sum \beta^t U(\hat{C}_{1t}, \hat{C}_{2t})}{\sum \beta^t U(\hat{C}^*_{1t}, \hat{C}^*_{2t})} = \frac{(1-\alpha)^\gamma \lambda^\gamma}{\alpha^\gamma (1-\lambda)^\gamma} \tag{4.4.5}$$

which implies that

$$\lambda = \frac{\alpha \hat{\pi}^{\frac{1}{\gamma}}}{\alpha \hat{\pi}^{\frac{1}{\gamma}} + (1-\alpha)(\hat{\pi}^*)^{\frac{1}{\gamma}}} \tag{4.4.6}$$

Therefore, if we are given a pair of equilibrium payoffs $(\hat{\pi}, \hat{\pi}^*)$, we can uncover the sequence of consumptions that are implied by these payoffs. Hence we can characterize the outcome of the game by the equilibrium payoffs.

The subgame perfect equilibrium in this game can be derived by using the similar arguments as in Sutton (1986). Let $\bar{u}(> 0)$ be the per period utility a consumer obtains under the competitive steady state. $\bar{u}$ is the same for both countries since the two countries have the same per capita consumption level under competitive equilibrium. If the world stayed in the competitive equilibrium forever, the payoff received by each country would be $\sum_{t=0}^{\infty} \beta^t \bar{u} = \frac{1}{1-\beta}\bar{u}$. $W(\tau_c) > \frac{1}{1-\beta}\bar{u}$ by the definition of $W(\tau_c)$. For the simplicity of notation, we define $W \equiv W(\tau_c)$.

Let π_m be the maximum payoff H will obtain in any subgame perfect equilibrium. Therefore, π_m is also the maximum payoff H will obtain in the subgame perfect equilibrium that starts from period 2. It is obvious that π_m must be greater than the payoff received from staying in competitive equilibrium. In other words, $\pi_m > \frac{1}{1-\beta}\bar{u}$.

Now consider period 1. If H rejects a proposal made by F, he obtains $\bar{u}$ in period 1, plus $\beta\pi_m$ at maximum in the future. Hence, any proposal that gives H a payoff higher than $\bar{u} + \beta\pi_m$ will certainly be accepted, which implies that the minimum payoff F will obtain in period 1 is $v^*(\bar{u} + \beta\pi_m)$.

In period 0, F will reject any proposal that offers him a payoff smaller than $\bar{u} + \beta v^*(\bar{u} + \beta\pi_m)$, which means that the maximum payoff H will obtain is $v^{*-1}[\bar{u} + \beta v^*(\bar{u} + \beta\pi_m)]$. Therefore, by the definition of π_m,

$$\pi_m = v^{*-1}[\bar{u} + \beta v^*(\bar{u} + \beta\pi_m)] \qquad (4.4.7)$$

Alternatively,

$$v^*(\pi_m) = \bar{u} + \beta v^*(\bar{u} + \beta\pi_m) \qquad (4.4.8)$$

It can be shown that (4.4.8) holds when π_m is defined instead as the minimum payoff H receives in a subgame perfect equilibrium. Therefore, (4.4.8) characterizes the subgame perfect equilibrium in this game.

Proposition 4.4.2. *Assume (A4.1)–(A4.5). The unique subgame perfect equilibrium in this game is characterized by*

$$(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{\gamma} - \beta[W^{\frac{1}{\gamma}} - \alpha(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}}]^{\gamma} = (1-\alpha)^{\gamma}\bar{u} \qquad (4.4.9)$$

and $\hat{\pi} \geq \frac{1}{1-\beta}\bar{u}$.

Proof: (4.4.9) is obtained by rewriting (4.4.8) using (4.3.24). Define the left-hand-side of (4.4.9) as $F(\pi)$. For $\pi \geq \frac{1}{1-\beta}\bar{u}$,

$$F'(\pi) = -\frac{\alpha\pi^{\frac{1-\gamma}{\gamma}}}{(W^{\frac{1}{\gamma}} - \alpha\pi^{\frac{1}{\gamma}})^{1-\gamma}} + \frac{\alpha\beta^2(\bar{u} + \beta\pi)^{\frac{1-\gamma}{\gamma}}}{[W^{\frac{1}{\gamma}} - \alpha(\bar{u} + \beta\pi)^{\frac{1}{\gamma}}]^{1-\gamma}} < 0 \qquad (4.4.10)$$

Since

$$F(\frac{1}{1-\beta}\bar{u}) = (1-\beta)(W^{\frac{1}{\gamma}} - \alpha(\frac{1}{1-\beta}\bar{u})^{\frac{1}{\gamma}})^{\gamma} > (1-\alpha)^{\gamma}\bar{u} \qquad (4.4.11)$$

and

$$F(\frac{W}{\alpha^\gamma}) = -\beta W[1-(\frac{\alpha^\gamma \bar{u}}{W}+\beta)^{\frac{1}{\gamma}}]^\gamma < 0 < (1-\alpha)^\gamma \bar{u}, \qquad (4.4.12)$$

there exists a unique $\hat{\pi}$ that satisfies (4.4.9) for $\hat{\pi} \geq \frac{1}{1-\beta}\bar{u}$.[(2)]

Q.E.D.

4.5. Negotiation: Further Analysis

The game presented in Section 4.4 differs from the standard Rubinstein bargaining game in two aspects. First, the two players are two countries with possibly different population sizes rather than two individuals. Second, the *status quo* yields positive payoffs to both players. This section is devoted to the investigation of the question that how these differences affect the bargaining solution. We shall start with the analysis on the special case where the utility function is homogenous of degree one ($\gamma = 1$), followed by an intuitive discussion of the results. We then derive the analogous results for the more general case $\gamma \in (0, 1]$.

If $\gamma = 1$, the subgame perfect equilibrium can be solved explicitly from (4.4.9), together with (4.3.24):

$$\hat{\pi} = \frac{W}{\alpha(1+\beta)} + \frac{\alpha(1+\beta)-1}{\alpha(1-\beta^2)}\bar{u} \qquad (4.5.1)$$

$$\hat{\pi}^* = \frac{\beta W}{(1-\alpha)(1+\beta)} - \frac{\alpha(1+\beta)-1}{(1-\alpha)(1-\beta^2)}\bar{u} \qquad (4.5.2)$$

In the standard Rubinstein bargaining game with identical time discount factors and linear bargaining frontier, the player that moves first obtains a larger share of the "pie" than the other player. In this model, however, it is not always the case. From (4.5.1) and (4.5.2) one can verify that when $\gamma = 1$, $\hat{\pi} > W > \hat{\pi}^*$ if $\alpha < \frac{1}{1+\beta}$; $\hat{\pi} < W < \hat{\pi}^*$ if $\alpha > \frac{1}{1+\beta}$; and $\hat{\pi} = \hat{\pi}^* = W$ if $\alpha = \frac{1}{1+\beta}$. The equilibrium per capita welfare of a country depends on her relative population size. In fact, when $\gamma = 1$,

$$\frac{d\hat{\pi}}{d\alpha} = -\frac{1}{\alpha^2(1+\beta)}(W - \frac{\bar{u}}{1-\beta}) < 0. \qquad (4.5.3)$$

$$\frac{d\hat{\pi}^*}{d\alpha} = \frac{\beta}{(1-\alpha)^2(1+\beta)}(W - \frac{\bar{u}}{1-\beta}) > 0. \tag{4.5.4}$$

In other words, the per capita welfare of a country is decreasing in her relative population size.

One might wonder what will happen to the aggregate welfare of a country as the relative population size changes.

$$\frac{d(\alpha\hat{\pi})}{d\alpha} = \frac{\bar{u}}{1-\beta} > 0 \tag{4.5.5}$$

$$\frac{d((1-\alpha)\hat{\pi}^*)}{d\alpha} = -\frac{\bar{u}}{1-\beta} < 0 \tag{4.5.6}$$

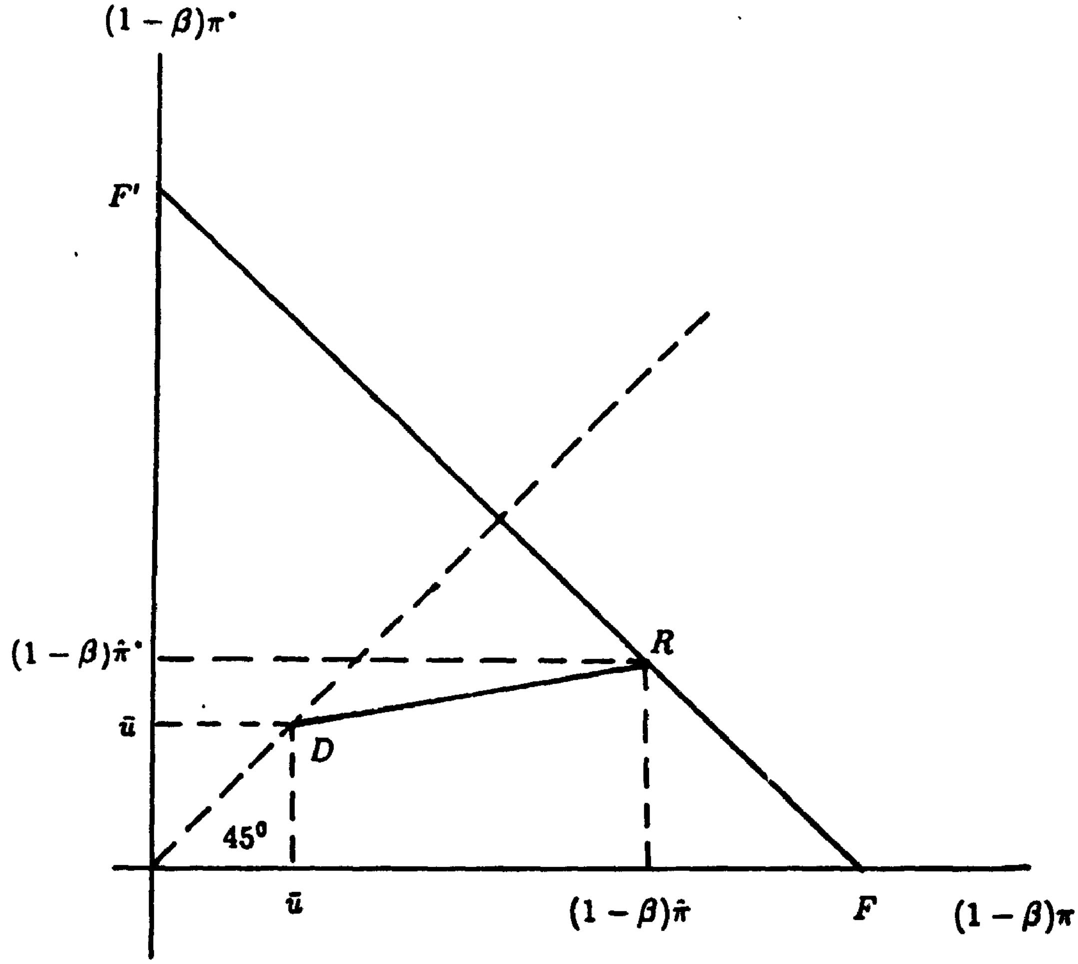
Therefore, the aggregate welfare of a country is increasing in her relative population size.

In the standard Rubinstein game, the disagreement point is (0, 0). In this model, however, a planner obtains a positive $\bar{u}$ for the period if no agreement is reached. While $\bar{u}$ is calculated endogenously from the competitive equilibrium of the model, it is exogenous to the bargaining game in question. Therefore, we can perform comparative statics on (4.5.1) and (4.5.2) to find out how $\bar{u}$ affects equilibrium outcomes.

$$\frac{d\hat{\pi}}{d\bar{u}} = \frac{\alpha(1+\beta)-1}{\alpha(1-\beta^2)} \begin{cases} > 0, & \text{if } \alpha > \frac{1}{1+\beta}; \\ = 0, & \text{if } \alpha = \frac{1}{1+\beta}, \\ < 0, & \text{if } \alpha < \frac{1}{1+\beta}. \end{cases} \tag{4.5.7}$$

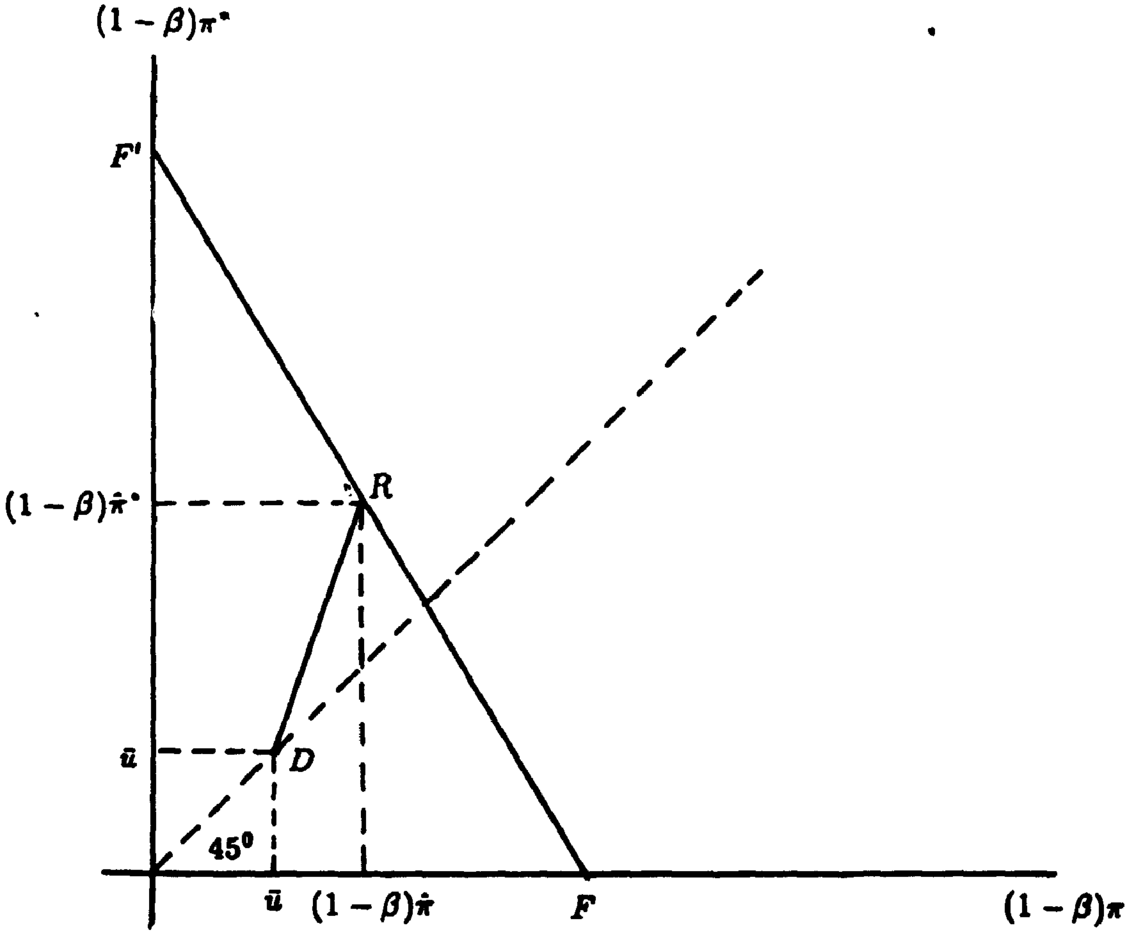
That is, Country H's payoff is increasing in $\bar{u}$ if $\alpha > \frac{1}{1+\beta}$ and is decreasing in $\bar{u}$ if $\alpha < \frac{1}{1+\beta}$. The reverse is true for Country F.

The above results are illustrated in Figures 4.5.1–4.5.6[3]. In Figure 4.5.1, $(1-\beta)\pi$ and $(1-\beta)\pi^*$ are the average per period payoffs measured in terms of per capita welfare of H and F, respectively. FF' is the welfare possibility frontier. D is the disagreement point. R is the bargaining solution. The slope of DR is $\frac{\alpha\beta}{1-\alpha}$. In Figure 4.5.1 DR is below the 45^0 line, which corresponds to the case $\alpha < \frac{1}{1+\beta}$. If $\alpha > \frac{1}{1+\beta}$, DR is above the 45^0 line, as illustrated in Figure 4.5.2.



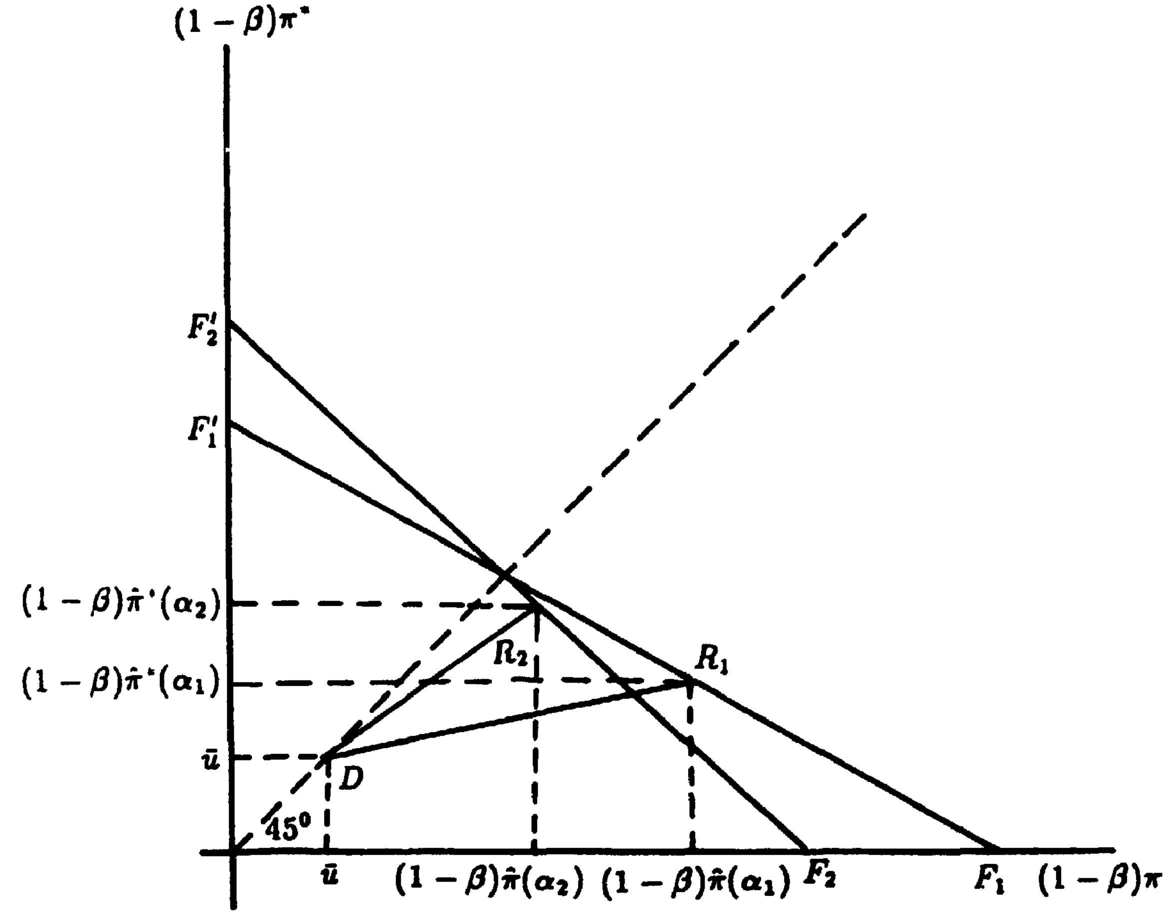
$\alpha < \frac{1}{1+\beta}$

FIGURE 4.5.1.



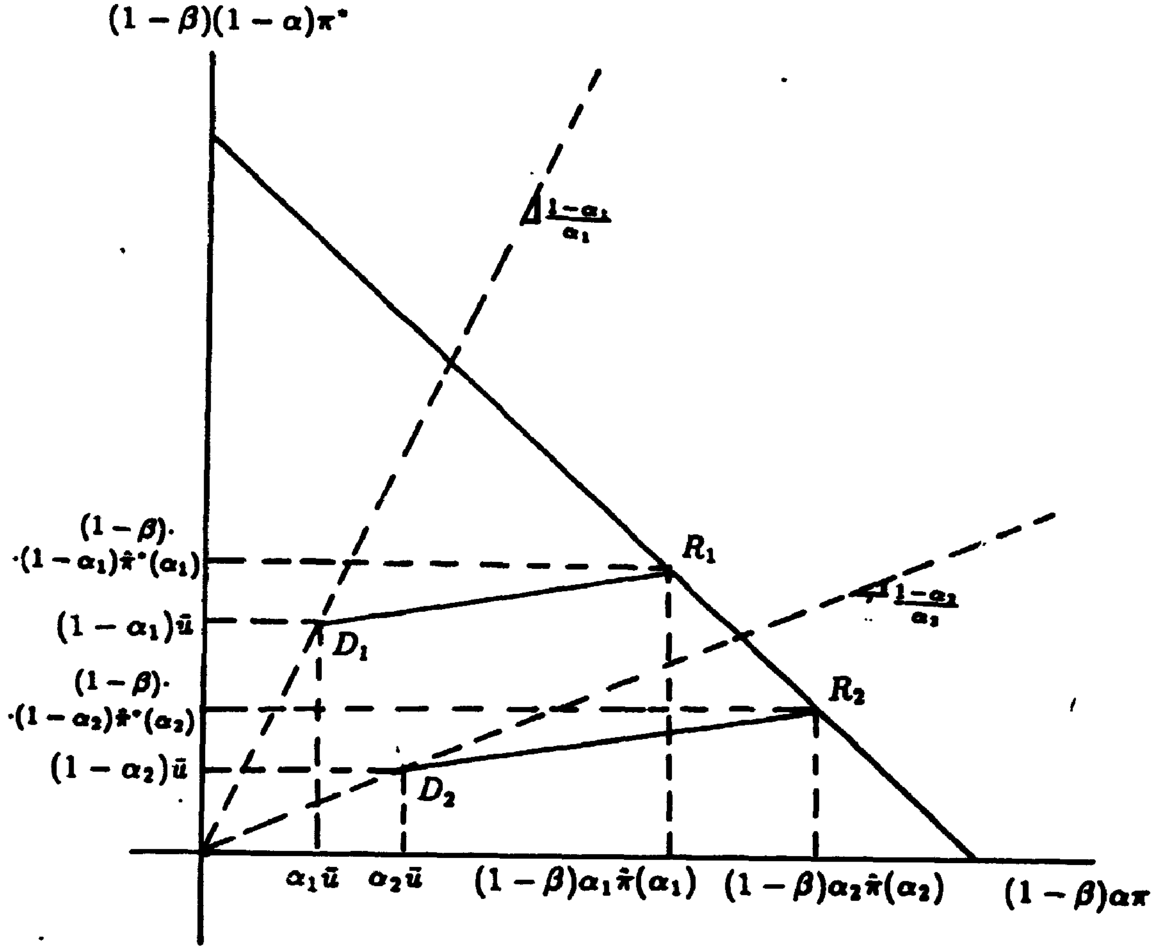
$\alpha > \frac{1}{1+\beta}$

FIGURE 4.5.2.



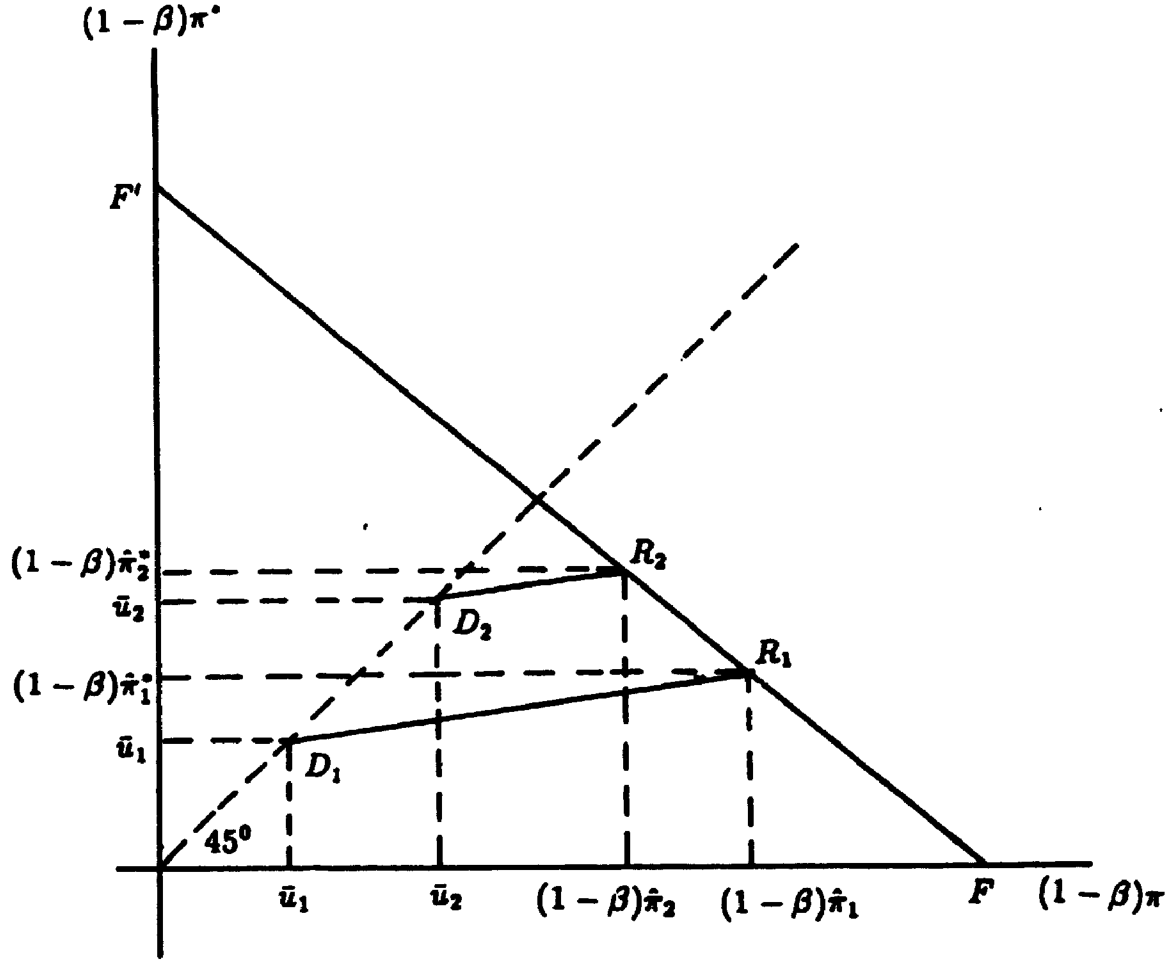
$\alpha_1 < \alpha_2$

FIGURE 4.5.3.



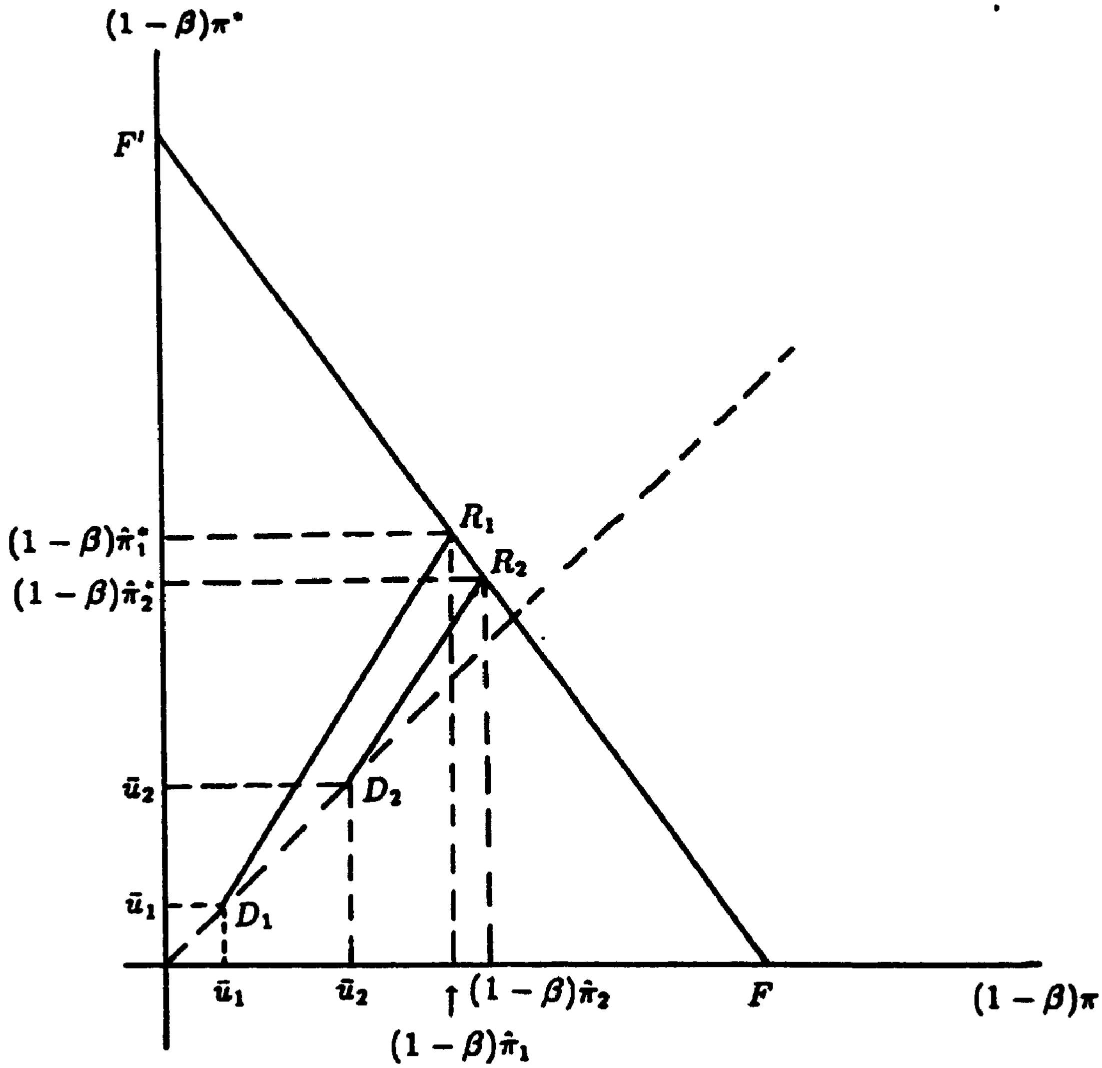
$\alpha_1 < \alpha_2$

FIGURE 4.5.4.



$\bar{u}_1 < \bar{u}_2 \qquad \alpha < \frac{1}{1+\beta}$

FIGURE 4.5.5.



$\bar{u}_1 < \bar{u}_2 \qquad \alpha > \frac{1}{1+\beta}$

FIGURE 4.5.6.

Figure 4.5.3 demonstrates the comparative statics of an increase in the value of α. As the value of α is increased from α_1 to α_2, the bargaining frontier shifts from F_1F_1' to F_2F_2'. The slope of DR becomes steeper as well. As a result, $(1-\beta)\hat{\pi}$ decreases while $(1-\beta)\hat{\pi}^*$ increases. Figure 4.5.4 illustrates the same comparative statics in the space of $((1-\beta)\alpha\pi,\ (1-\beta)(1-\alpha)\pi^*)$, the space of the average per period welfare measured in terms of the aggregate welfare of the two countries. As α changes, the welfare possibility frontier does not shift but the disagreement point moves. $(1-\beta)\alpha\hat{\pi}$ increases while $(1-\beta)(1-\alpha)\hat{\pi}^*$ decreases in response to an increase in α.

Figures 4.5.5 and 4.5.6 illustrate the comparative statics with respect to $\bar{u}$. As $\bar{u}$ changes from $\bar{u}_1$ to $\bar{u}_2$, the disagreement point moves from D_1 to D_2. As a consequence the bargaining solution moves from R_1 to R_2. Figure 4.5.5 is drawn under the condition $\alpha < \frac{1}{1+\beta}$, in which case an increase in $\bar{u}$ reduces $(1-\beta)\hat{\pi}$ but raises $(1-\beta)\hat{\pi}^*$. Figure 4.5.6 is drawn for the case $\alpha > \frac{1}{1+\beta}$ where the reverse is true.

Next we consider the more general case where $\gamma \in (0,\ 1]$. The proofs of the following results are straightforward but involve long mathematical expressions. Hence the proofs are relegated to Appendix III.

Define $d \equiv W - \frac{1}{1-\beta}\bar{u}$. $d > 0$ by construction. For the ease of exposition, define two new variables: $x \equiv 1 + \frac{1-\beta}{\beta}\frac{d}{W}$ and $y \equiv 1 - (1-\beta)\frac{d}{W}$. It is obvious that $x > 1$ and $0 < y < 1$. Define

$$\tilde{\alpha} \equiv \frac{x^{\frac{1}{\gamma}} - 1}{x^{\frac{1}{\gamma}} - y^{\frac{1}{\gamma}}} \tag{4.5.8}$$

Lemma 4.5.1. *Assume (A4.1)–(A4.5).* $\frac{1}{2} < \tilde{\alpha} < 1$.

Proof: See Appendix III.

Proposition 4.5.2. *Assume (A4.1)–(A4.5).* $\hat{\pi} > W > \hat{\pi}^*$ *if* $\alpha < \tilde{\alpha}$. $\hat{\pi} < W < \hat{\pi}^*$ *if* $\alpha > \tilde{\alpha}$. $\hat{\pi} = \hat{\pi}^* = W$ *if* $\alpha = \tilde{\alpha}$.

Proof: See Appendix III.

Proposition 4.5.3. *Assume (A4.1)–(A4.5). The equilibrium per capita payoff of Country H, $\hat{\pi}$, is a decreasing function of her relative population size, α.*

Proof: See Appendix III.

It has been shown earlier in this section that $\hat{\pi}^*$ is increasing in α if $\gamma = 1$. The extension of this result to the case $\gamma \in (0, 1)$, however, requires additional restrictions.

Proposition 4.5.4. *Assume (A4.1)–(A4.5). Consider the case $\gamma \in (0, 1)$. The equilibrium per capita payoff of Country F, $\hat{\pi}^*$, is a decreasing function of her relative population size $(1 - \alpha)$ if $\bar{u} < (1 - \beta)\alpha^{1-\gamma}W$.*

Proof: See Appendix III.

Proposition 4.5.5. *Assume (A4.1)–(A4.5). The equilibrium aggregate payoff of Country H, $\alpha\hat{\pi}$, is increasing in α while the equilibrium aggregate payoff of Country F, $(1 - \alpha)\hat{\pi}^*$, is decreasing in α.*

Proof: See Appendix III.

Define a function

$$\Delta(\alpha) \equiv \beta\alpha(\bar{u} + \beta\hat{\pi})^{\frac{1-\gamma}{\gamma}} - (1 - \alpha)^{\gamma}[W^{\frac{1}{\gamma}} - \alpha(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}}]^{1-\gamma} \tag{4.5.9}$$

Lemma 4.5.6. *Assume (A4.1)–(A4.5). There exists $\hat{\alpha} \in (0, 1)$ such that $\Delta(\hat{\alpha}) =$ 0.*

Proof: See Appendix III.

Lemma 4.5.7. *Assume (A4.1)–(A4.5). $\hat{\alpha}$ is unique. Furthermore, $\hat{\alpha} > \frac{1}{2}$.*

Proof: See Appendix III.

Proposition 4.5.8. *Assume (A4.1)–(A4.5). If $\alpha < \hat{\alpha}$, then $\hat{\pi}$ is decreasing in $\bar{u}$ while $\hat{\pi}^*$ is increasing in $\bar{u}$. If $\alpha > \hat{\alpha}$, then $\hat{\pi}$ is increasing in $\bar{u}$ while $\hat{\pi}^*$ is decreasing in $\bar{u}$. If $\alpha = \hat{\alpha}$, then $\hat{\pi}$ and $\hat{\pi}^*$ are independent of $\bar{u}$.*

Proof: See Appendix III.

When $\gamma = 1$, $\hat{\alpha} = \tilde{\alpha} = \frac{1}{1+\beta}$. Is it true that $\hat{\alpha} = \tilde{\alpha}$ for all $\gamma \in (0,\ 1]$?

Proposition 4.5.9. *$\tilde{\alpha} < \hat{\alpha}$ if $\gamma \in (0,\ 1)$. $\tilde{\alpha} = \hat{\alpha}$ if $\gamma = 1$.*

Proof: See Appendix III.

In this game, the standard Rubinstein bargaining game is equivalent to the case where $\alpha = \frac{1}{2}$ (two identical players) and $\bar{u} = 0$. It is easy to verify that in this case $\hat{\pi} > \hat{\pi}^*$, a standard result of the Rubinstein bargaining game. However, as has been demonstrated in this section, this result does not always hold for other values of α and $\bar{u}$. An increase in the relative population size of the country that makes the first move in the bargaining process (Country H) offsets her first mover's advantage. The first mover's advantage will be completely offset if Country H's population size exceeds $\tilde{\alpha}$.

With a positive $\bar{u}$, the gain from an agreement for both countries decreases as $\bar{u}$ increases. One might have expected that the first mover's advantage of Country H diminishes as $\bar{u}$ increases. Indeed, in the case where $\alpha = \frac{1}{2}$ and $\bar{u} > 0$, while it is still true that $\hat{\pi} > \hat{\pi}^*$, $\hat{\pi}$ is decreasing in $\bar{u}$. However, as demonstrated in Proposition 4.5.8, this is not always true in the case of unequal population sizes. $\frac{d\hat{\pi}}{d\bar{u}} < 0$ is true only if the relative population size of Country H does not exceed some critical value ($\alpha < \hat{\alpha}$).

4.6. Decentralization

In Section 4.4, the negotiation outcome is derived under the assumption that the two planners have the authority to choose consumption and production plans for their countries. The purpose of this section is to demonstrate that how the governments in the two countries can implement such an agreement in decentralized economies.

The agreement negotiated by the two planners specifies a sequence of consumption and production plans

$$\{\hat{C}_{1t}, \hat{C}_{2t}, \hat{C}^*_{1t}, \hat{C}^*_{2t}, \hat{L}_{1t}, \hat{L}_{2t}, \hat{S}_{1t} + \hat{S}^*_{1t}, \hat{S}_{2t} + \hat{S}^*_{2t}\}_{t=0}^{\infty}.$$

Since the two countries have identical production technology and identical climate, the location of production can be chosen arbitrarily. The implementation of the agreement in competitive economies is to device a system of tax and international transfers so that the competitive equilibrium generates the same sequence of outputs and consumption for the two countries as requir ,y the agreement.

Consider a tax/transfer system where there is a worldwide pollution tax on the unit cost of the manufacturing production and a country-specific lump-sum tax (transfer) on (to) consumers. The total revenue from the pollution tax is distributed equally among the world residents in a lump-sum fashion. The revenue from the lump-sum tax on the consumers of one country is used to finance the lump-sum transfer to the consumers in the other country. Let η_t be the period t world pollution tax rate. Set

$$\begin{aligned}\eta_t = & \frac{b\beta(1-c)G'(t+1)}{a(\tau_t)G'(t)} \frac{U_1(\hat{S}_{1t+1} + \hat{S}^*_{1t+1}, \hat{S}_{2t+1} + \hat{S}^*_{2t+1})}{U_1(\hat{S}_{1t} + \hat{S}^*_{1t}, \hat{S}_{2t} + \hat{S}^*_{2t})} \cdot \\ & \cdot \left[\frac{U_2(\hat{S}_{1t+1} + \hat{S}^*_{1t+1}, \hat{S}_{2t+1} + \hat{S}^*_{2t+1})}{U_1(\hat{S}_{1t+1} + \hat{S}^*_{1t+1}, \hat{S}_{2t+1} + \hat{S}^*_{2t+1})} - \frac{a(\tau_{t+1})}{b}\right] \\ & - \frac{\beta U_1(\hat{S}_{1t+1} + \hat{S}^*_{1t+1}, \hat{S}_{2t+1} + \hat{S}^*_{2t+1})}{U_1(\hat{S}_{1t} + \hat{S}^*_{1t}, \hat{S}_{2t} + \hat{S}^*_{2t})} a'(\tau_{t+1})(1 - \hat{L}_{2t+1}) \frac{b}{a(\tau_t)G'(t)}.\end{aligned} \tag{4.6.1}$$

Define ρ_t and ρ^*_t as the lump-sum tax (transfer) in Country H and Country F, respectively.

$$\rho_t = (\hat{S}_{1t} + \hat{S}^*_{1t} - \hat{C}_{1t}) + \frac{U_2(\hat{C}_{1t}, \hat{C}_{2t})}{U_1(\hat{C}_{1t}, \hat{C}_{2t})}(\hat{S}_{2t} + \hat{S}^*_{2t} - \hat{C}_{2t}) \tag{4.6.2}$$

and ρ^*_t is defined in the same way as ρ_t:

$$\rho^*_t = (\hat{S}_{1t} + \hat{S}^*_{1t} - \hat{C}^*_{1t}) + \frac{U_2(\hat{C}^*_{1t}, \hat{C}^*_{2t})}{U_1(\hat{C}^*_{1t}, \hat{C}^*_{2t})}(\hat{S}_{2t} + \hat{S}^*_{2t} - \hat{C}^*_{2t}) \tag{4.6.3}$$

The sequences of η_t, ρ_t and ρ^*_t can be calculated using the consumption and production sequences specified by the agreement. η_t is the same in both countries so that the pollution tax will not cause price disparities across countries. It is easy to verify that

$$\alpha\rho_t + (1-\alpha)\rho^*_t = 0 \tag{4.6.4}$$

Proposition 4.6.1. *Assume (A4.1)-(A4.5). The agreement by the two central planners can be implemented in competitive economies by the tax/transfer system specified above.*

Proof: The idea behind this proof is to show that given the tax/transfer system, the competitive equilibrium yields the same optimization conditions as the world planner's problem. We shall prove the result in terms of the competitive equilibrium in Country H. The competitive equilibrium in Country F is completely symmetric. Let w_t denote the period t wage rate in Country H and P_t the period t relative price of the manufactured good in terms of the consumption good. Since there is no capital good in this model, the firms' objective is to maximize their profits in each period. The optimization problem of a representative firm in the agricultural sector in Country H is:

$$\max_{\phi_{1t}} a(\tau_t)\phi_{1t} - w_t\phi_{1t} \tag{4.6.5}$$

Similarly, the optimization problem of a representative firm in the manufacturing sector is

$$\max_{\phi_{2t}} P_t b\phi_{2t} - (1+\eta_t)w_t\phi_{2t} \tag{4.6.6}$$

Since both goods are produced in the world in equilibrium, we have

$$P_t b = (1 + \eta_t)a(\tau_t) \tag{4.6.7}$$

A representative consumer in Country H receives the rebate of the pollution tax revenue on the one hand, and faces a lump-sum tax (transfer) ρ_t on the other. Let μ_t be the pollution tax rebate he receives in period t. Let Z_t denote the lending he made in period t. Z_t is measured in the unit of the period t agricultural good. The interest rate prevailing between period $t-1$ and period t is denoted by r_t. His optimization problem can be written as:

$$\max_{\{C_{1t}, C_{2t}, Z_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_{1t}, C_{2t}) \tag{4.6.8}$$

subject to

$$C_{1t} + P_t C_{2t} + Z_t = w_t + \mu_t - \rho_t + (1 + r_t)Z_{t-1} \tag{4.6.9}$$

(4.6.8)-(4.6.9) gives the standard optimization condition

$$P_t = \frac{U_2(C_{1t}, C_{2t})}{U_1(C_{1t}, C_{2t})} \tag{4.6.10}$$

$$U_1(C_{1t}, C_{2t}) = \beta(1 + r_{t+1})U_1(C_{1t+1}, C_{2t+1}) \tag{4.6.11}$$

One can derive analogous first-order conditions for Country F. Since the two countries face the same prices P_t,

$$\frac{U_2(C_{1t}, C_{2t})}{U_1(C_{1t}, C_{2t})} = \frac{U_2(S_{1t} + S_{1t}^*, S_{2t} + S_{2t}^*)}{U_1(S_{1t} + S_{1t}^*, S_{2t} + S_{2t}^*)} \tag{4.6.12}$$

and

$$\frac{U_1(C_{1t+1}, C_{2t+1})}{U_1(C_{1t}, C_{2t})} = \frac{U_1(S_{1t+1} + S_{1t+1}^*, S_{2t+1} + S_{2t+1}^*)}{U_1(S_{1t} + S_{1t}^*, S_{2t} + S_{2t}^*)} \tag{4.6.13}$$

Substitute (4.6.1), (4.6.10), and (4.6.12)-(4.6.13) into (4.6.7) and re-arrange:

$$\begin{aligned} G'(t)[\frac{a(\tau_t)}{b} - \frac{U_2(C_{1t}, C_{2t})}{U_1(C_{1t}, C_{2t})}] &= \beta\frac{U_1(C_{1t+1}, C_{2t+1})}{U_1(C_{1t}, C_{2t})}a'(\tau_{t+1})(1 - L_{2t+1}) \\ &+ \beta(1 - c)G'(t+1)\frac{U_1(C_{1t+1}, C_{2t+1})}{U_1(C_{1t}, C_{2t})}[\frac{a(\tau_{t+1})}{b} - \frac{U_2(C_{1t+1}, C_{2t+1})}{U_1(C_{1t+1}, C_{2t+1})}] \end{aligned} \tag{4.6.14}$$

(4.6.14) is equivalent to the world planner's optimization condition (4.3.19). Therefore, the competitive equilibrium with the tax/transfer system will generate the optimal production and temperature sequences.

Next we show that the tax system generates the consumption sequence for Country H as specified by the agreement. In equilibrium, $\mu_t = \eta_t w_t L_{2t}$. Suppose that there is no borrowing and lending in equilibrium, ie, $Z_t = 0$. Substitute the equilibrium value of μ_t and ρ_t into (4.6.9), we have:

$$C_{1t} + P_t C_{2t} = \hat{C}_{1t} + P_t \hat{C}_{2t} \tag{4.6.15}$$

It is easy to verify that $\{\hat{C}_{1t}, \hat{C}_{2t}, \hat{C}^*_{1t}, \hat{C}^*_{2t}\}_{t=0}^{\infty}$ satisfies the first-order conditions (4.6.10)-(4.6.11) for both countries given appropriately chosen prices and interest rates $\{P_t, r_t\}_{t=0}^{\infty}$. Therefore, $\{\hat{C}_{1t}, \hat{C}_{2t}, \hat{C}^*_{1t}, \hat{C}^*_{2t}\}_{t=0}^{\infty}$ can be supported as an equilibrium consumption sequence if borrowing and lending among consumers is prohibited.

Finally, we prove that under the tax/transfer system specified above, $Z_t = 0$. In other words, there is no borrowing and lending among consumers in a competitive equilibrium. In equilibrium there is no borrowing and lending within a country since all consumers in the country are identical and have the same income level. We need only to prove that there is no international borrowing and lending in equilibrium.

In equilibrium, the two countries face the same interest rates r_t. (4.6.11) and the homogeneity of utility function imply

$$\frac{C_{1t}}{C_{1t+i}} = \frac{C^*_{1t}}{C^*_{1t+i}}, \quad i = 1, 2, \cdots. \tag{4.6.16}$$

Notice that w_t, μ_t are determined by the production side of the economy and ρ_t is specified by the agreement between the two countries. They are not influenced by the amount of borrowing and lending among consumers.

Consider a sequence of consumption that involves non-zero borrowing and lending among countries, $\{\tilde{C}_{1t}, \tilde{C}_{2t}, \tilde{C}^*_{1t}, \tilde{C}^*_{2t}\}_{t=0}^{\infty}$. Let period s be the earliest period in which borrowing and lending occurs. Without any loss of generality, assume that $\tilde{Z}_s > 0 > \tilde{Z}_s^*$, ie, H lends to F. From (4.6.9) we know that the borrowing in period s will be used to finance F's consumption in period s. Then $\tilde{C}_{js} < \hat{C}_{js}$ and $\tilde{C}_{js}^* > \hat{C}_{js}^*$ $(j = 1, 2)$. Since F has to pay back the debt in some future period $s + i$ $(i \geq 1)$, $\tilde{C}_{js+i} > \hat{C}_{js+i}$ and $\tilde{C}_{js+i}^* < \hat{C}_{js+i}^*$ $(j = 1, 2)$. Recall from (4.4.3)-(4.4.4) that

$$\frac{\hat{C}_{1t}}{\hat{C}_{1t+i}} = \frac{\hat{C}_{1t}^*}{\hat{C}_{1t+i}^*}, \quad i = 1, 2, \cdots \tag{4.6.17}$$

by construction. Then

$$\frac{\tilde{C}_{1t}}{\tilde{C}_{1t+i}} < \frac{\tilde{C}_{1t}^*}{\tilde{C}_{1t+i}^*} \tag{4.6.18}$$

which violates (4.6.16). Therefore, there is no borrowing and lending in equilibrium.

Q.E.D.

Proposition 4.6.1 implies that, instead of negotiating the consumption and production plans for the two countries, the two governments can negotiate a tax/transfer system that generates exactly the same outcome.

Proposition 4.6.2. *$\rho_t > 0 > \rho_t^*$ if $\hat{\pi} < W < \hat{\pi}^*$, and $\rho_t < 0 < \rho_t^*$ if $\hat{\pi} > W > \hat{\pi}^*$.*

Proof: By Proposition 4.4.1, equations (4.3.8)-(4.3.11) and the definitions of π, π^* and W,

$$\hat{C}_{jt} < \hat{S}_{jt} + \hat{S}_{jt}^* < \hat{C}_{jt}^*, \quad (j = 1, 2) \tag{4.6.19}$$

if $\hat{\pi} < W < \hat{\pi}^*$; and the reverse is true if $\hat{\pi} > W > \hat{\pi}^*$. The result follows from (4.6.2) and (4.6.3).

Q.E.D.

In the tax/transfer regime discussed above, the pollution tax revenue from the two countries is pooled and is distributed equally among the world residents. In this process, some transferring of the tax revenue from one country to the other may occur. To see this consider the Country H's government budget of the pollution tax. The pollution tax revenue is equal to $\eta_t w_t \phi_{2t}$ while the payment to her citizens is

$\alpha\mu_t = \eta_t w_t \alpha L_{2t}$. Therefore, if Country H's share of the output of the manufactured good is larger than her share in the world population, ie, $\frac{\phi_{2t}}{L_{2t}} > \alpha$, part of the pollution tax revenue will be transferred to Country F to finance the payment to her consumers.

Therefore, in equilibrium, in general we will observe two kinds of international transfers. First, the transfers that are made due to the disproportional distribution of the manufacturing productions. We will observe that the country that produces more than her share of manufacturing output and hence emits more than her share of carbon dioxide transfers part of her pollution tax revenue to finance the compensation in the other country. Second, the transfers that are made due to the asymmetry in bargaining power. The gain from international cooperation is not shared equally among the two countries as a result of asymmetry in the bargaining procedure and in relative population sizes.

4.7. Conclusions

In this chapter, the global optimal time path of consumption, production and temperature is characterized. Furthermore, it is demonstrated that the global optimality can be achieved under competitive equilibrium through a binding agreement between two countries that specifies a system of taxes and transfers.

One important issue discussed in most of the existing literature on global environmental problems is the side payments between countries. In the existing analyses, side payments are made either for compensating the victim of pollution for damages or for compensating the polluter for not polluting (See, for example, Barrett, 1990, Mäler, 1990). In this chapter, it is shown that in general international transfers from one country to the other will be observed in equilibrium. Not all of these transfers, however, are made by the heavy-polluting country for the purpose of compensating the damages in the other country. Part of these transfers are made purely due to the asymmetry in bargaining power. Therefore, the observed international transfers are not necessarily made for the "right" reason, ie, for compensating the victim country.

Economists have pointed out many difficulties associated with controlling the global warming (Barrett 1990, Nordhaus 1990a). Among them, the two most important ones are: (1) the cooperation among countries; and (2) the uncertainty and the lack of information about future climate changes. In this chapter, it is demonstrated that if the governments have perfect information, the cooperation problem may be solved by negotiating a binding agreement on an international tax/transfer system. In the real world, of course, the governments do not have perfect information. Given the current state of scientific and economic research on the global warming, the governments do not have enough information on the possible losses (or gains) that will arise from future climate changes. Therefore, it is the lack of information that is preventing governments from cooperating on the global warming issue. The key to solving the global warming problem is information rather than cooperation.

The Montreal Protocol for the protection of the ozone layer offers an example that supports our arguments that international cooperation can be achieved if sufficient information is available. In contrast to the uncertainties with regard to the existence and the effects of the global warming, the evidence on the depletion of ozone layer by CFCs is clear and convincing. The 1985 report by the British Antarctic Survey showed that massive ozone depletion occurs over the Antarctic each spring. As a result of this disturbing discovery, 24 nations met in Montreal and signed the so-called Montreal Protocol in 1987. Since then, as more conclusive evidence becomes available, the urgency of protecting the ozone layer becomes more apparent. In June 1990, representatives from 75 countries gathered in London and signed an accord that dramatically strengthened the Montreal Protocol.

In this chapter, the non-cooperative bargaining theory is employed in deriving the agreement between the two countries. The Rubinstein bargaining solution is derived from a well-specified economic environment. In the process, some interesting new results for the non-cooperative bargaining theory are generated.

In the standard Rubinstein game, the players' time preference is the only factor that affects their bargaining power. In this model, because of its richer structure, in addition to the countries' time discount factors, the other factors, such as the relative size of population and the payoffs at *status quo*, also affect a country's bargaining power. While the first mover's advantage still exists, the country that moves first will not necessarily obtain a higher payoff in per capita terms than the other country because of the effects of other factors.

It is shown that the aggregate bargaining power of a country improves (in the sense that the aggregate payoff increases) as the relative population size of the country increases. Such an increase, however, is less than proportional to the augmentation in relative population size. As a consequence, the per capital payoff of a country decreases as the country's relative population size increases.

Another feature of this bargaining game is that the *status quo* point in this model generates non-zero payoffs to the players. Intuition suggests that an increase in the payoffs at the *status quo* for both countries should diminish the first mover's advantage because the second mover has less to lose from delaying an agreement. It is shown that this conjecture is not always true. The opposite is true if the relative population size of the country that makes the first move is larger than certain critical value.

Footnotes

(1) See Chapter 3 for conditions under which the world reaches a steady state under competitive equilibrium.

(2) We are unable to rule out the possibility that (4.4.9) has a solution for π in the interval $(0, \frac{1}{1-\beta}\bar{u})$ in the case where $\gamma < 1$. Such a solution, if exists, is obviously not a subgame perfect equilibrium.

(3) I am grateful to Ignatius Horstmann for his comments and suggestions on these diagrammatic illustrations.

Appendix I

Appendix to Chapter 1

In this appendix we present the detailed expressions of the derivatives that are used in the text of Chapter 1. The equation numbers used in this appendix correspond to their counterparts in the text.

$$\Delta(t) = (G_{zz}\beta V_{kk}(t+1) + G_{zz}P_t^2 U_{cc} + \beta V_{kk}(t+1)F_{1l}^2 U_{cc})\cdot$$
$$\cdot P_t U_c^2 (1 - \frac{k_{1t}l_{2t}}{l_{1t}k_{2t}})^2 F_{1kk}F_{2ll} > 0$$

is the Jacobian associated with the system of equations (1.3.9)-(1.3.12). Let

$$A \equiv P_t U_c^2 F_{1kk} F_{2ll} > 0,$$

$$Q \equiv G_{zz}\beta V_{kk}(1) + G_{zz}P_0^2 U_{cc} + \beta V_{kk}(1)F_{1l}^2 U_{cc} > 0.$$

$$\frac{\partial l_{10}}{\partial H} = \frac{1}{\Delta(0)} G_{zz}(\beta V_{kk}(1) + P_0^2 U_{cc})(1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})A \begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}. \end{cases} \quad (1.4.1)$$

$$\frac{\partial l_{20}}{\partial H} = \frac{1}{\Delta(0)}[G_{zz}\beta V_{kk}(1) + G_{zz}P_0^2 U_{cc}]\frac{k_{10}l_{20}}{l_{10}k_{20}}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A \begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \quad (1.4.2)$$

$$\frac{\partial k_{20}}{\partial H} = -\frac{\partial k_{10}}{\partial H} = \frac{1}{\Delta(0)}[G_{zz}\beta V_{kk}(1) + G_{zz}P_0^2 U_{cc}]\frac{k_{10}}{l_{10}}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \quad (1.4.3)$$

$$\frac{\partial l_{10}}{\partial K_0} = \frac{1}{\Delta(0)}[\frac{l_{20}}{k_{20}}Q + \beta V_{kk}(1)F_{1l}U_{cc}(F_{1k} + P_t(1-\delta))](\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \quad (1.4.4)$$

$$\frac{\partial k_{10}}{\partial K_0} = \frac{1}{\Delta(0)}[\frac{l_{20}}{k_{20}}Q + \beta V_{kk}(1)F_{1l}U_{cc}(F_{1k} + P_t(1-\delta))]\frac{k_{10}}{l_{10}}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \quad (1.4.5)$$

$$\frac{\partial l_{20}}{\partial K_0} = \frac{1}{\Delta(0)}[Q + \beta V_{kk}(1) F_{1l} U_{cc}(F_{1k} + P_t(1-\delta))\frac{k_{10}}{l_{10}}]\frac{l_{20}}{k_{20}}(1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})A$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.6}$$

$$\frac{\partial k_{20}}{\partial K_0} = \frac{1}{\Delta(0)}[Q + \beta V_{kk}(1) F_{1l} U_{cc}(F_{1k} + P_t(1-\delta))\frac{k_{10}}{l_{10}}](1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})A$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.7}$$

$$\frac{\partial(F_1(0) - C_0)}{\partial H}$$
$$= \frac{G_{zz}}{\Delta(0)}[F_{1k}(\beta V_{kk}(1) + P_0^2 U_{cc})\frac{k_{10}}{l_{10}} + F_{1l}P_0^2 U_{cc} + F_{1l}\beta V_{kk}(1)\frac{k_{10}l_{20}}{l_{10}k_{20}}](1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})A$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.10}$$

$$\frac{\partial(F_2(0) - I_0)}{\partial H}$$
$$= \frac{G_{zz}}{\Delta(0)}[F_{2k}\frac{k_{10}}{l_{10}}(\beta V_{kk}(1) + P_0^2 U_{cc}) + F_{2l}P_0^2 U_{cc} + F_{2l}\beta V_{kk}(1)\frac{k_{10}l_{20}}{l_{10}k_{20}}](\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases} \tag{1.4.11}$$

$$\begin{aligned} &\frac{\partial(F_1(0) - C_0)}{\partial K_0} \\ &= \frac{1}{\Delta(0)}\{F_{1l}[\frac{l_{20}}{k_?}Q + \beta V_{kk}(1)F_{1l}U_{cc}(F_{1k} + P_0(1-\delta))](\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A \\ &+ F_{1k}\frac{k_{10}l_{20}}{l_{10}k_{20}}U_{cc}(P_0^2 G_{zz} + \beta V_{kk}(1)F_{1l}^2)(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A \\ &+ F_{1k}\beta V_{kk}(1)(\frac{k_{10}}{l_{10}}U_{cc}F_{1l}F_{1k} + P_0(1-\delta))(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)A \\ &+ P_0(1-\delta)\beta V_{kk}(1)G_{zz}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A\} \end{aligned} \tag{1.4.12}$$
$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}} \text{ and } \delta = 1; \\ \gtrless 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}} \text{ and } \delta \in [0,\ 1); \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}. \end{cases}$$

$$\frac{\partial(F_2(0) - I_0)}{\partial K_0}$$

$$= \frac{1}{\Delta(0)}\{F_{2l}Q + \beta V_{kk}(1)F_{1l}U_{cc}(F_{1k} + P_0(1-\delta))\frac{k_{10}}{l_{10}}]\frac{l_{20}}{k_{20}}(1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})A$$

$$+ F_{2k}\beta V_{kk}(1)(G_{zz} + F_{1l}^2 U_{cc})(1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})A$$

$$+ F_{2k}U_{cc}\frac{k_{10}}{l_{10}}[\beta V_{kk}(1)F_{1l}(F_{1k} + P_0(1-\delta)) + P_0^2 G_{zz}\frac{l_{20}}{k_{20}}](1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})A \qquad (1.4.13)$$

$$+ (1-\delta)\beta V_{kk}(1)(G_{zz} + F_{1l}^2 U_{cc})(1 - \frac{k_{10}l_{20}}{l_{10}k_{20}})^2 A\}$$

$$\begin{cases} > 0, & \text{if } \frac{k_{10}}{l_{10}} < \frac{k_{20}}{l_{20}}; \\ < 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}} \text{ and } \delta = 1; \\ \gtrless 0, & \text{if } \frac{k_{10}}{l_{10}} > \frac{k_{20}}{l_{20}} \text{ and } \delta \in [0,\ 1). \end{cases}$$

$$\frac{\partial K_1}{\partial H} = \frac{1}{\Delta(0)}G_{zz}P_0U_{cc}F_{1l}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.14)$$

$$\frac{\partial K_1}{\partial K_0} = \frac{1}{\Delta(0)}G_{zz}P_0U_{cc}(F_{1k} + P_0(1-\delta))(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.15)$$

$$\frac{\partial(y_{10} + P_0 y_{20})}{\partial H} = \frac{1}{\Delta(0)}F_{1l}G_{zz}(\beta V_{kk}(1) + P_0^2 U_{cc})(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.21)$$

$$\frac{\partial C_0}{\partial H} = \frac{1}{\Delta(0)}\beta V_{kk}(1)F_{1l}G_{zz}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.22)$$

$$\frac{\partial Z_0}{\partial H} = \frac{1}{\Delta(0)}\beta V_{kk}(1)F_{1l}^2 U_{cc}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.23)$$

$$\frac{\partial(y_{10} + P_0 y_{20})}{\partial K_0} = \frac{1}{\Delta(0)}(F_{1k} + P_0(1-\delta))G_{zz}(\beta V_{kk}(1) + P_0^2 U_{cc})(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.24)$$

$$\frac{dC_0}{dK_0} = \frac{1}{\Delta(0)}\beta V_{kk}(1)(F_{1k} + P_0(1-\delta))G_{zz}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.25)$$

$$\frac{\partial Z_0}{\partial K_0} = \frac{1}{\Delta(0)}\beta V_{kk}(1)F_{1l}U_{cc}(F_{1k} + P_0(1-\delta))(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.26)$$

$$\frac{\partial(l_{10} + l_{20})}{\partial H} = \frac{1}{\Delta(0)}G_{zz}(\beta V_{kk}(1) + P_t^2 U_{cc})(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A > 0 \qquad (1.4.30)$$

$$\frac{\partial(\frac{l_{10}+l_{20}}{H})}{\partial H} =$$

$$\frac{1}{\Delta(0)H^2}[Z_t G_{zz}(\beta V_{kk}(1) + P_t^2 U_{cc}) - (l_{10} + l_{20})\beta V_{kk}(1) F_{1l}^2 U_{cc}(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A \gtrless 0. \qquad (1.4.31)$$

$$\frac{\partial(l_{10} + l_{20})}{\partial K_0} = -\frac{1}{\Delta(0)}\beta V_{kk}(1) F_{1l} U_{cc}(F_{1k} + P_0(1-\delta))(\frac{k_{10}l_{20}}{l_{10}k_{20}} - 1)^2 A < 0. \quad (1.4.32)$$

$$\begin{aligned} \frac{\partial k_{2t-1}}{\partial \Delta K_t} &= -\frac{\partial k_{1t-1}}{\partial \Delta K_t} \\ &= -\frac{1}{\Delta(t-1)}\beta V_{kk}(t) F_{1l} U_{cc} P_{t-1}\frac{k_{1t-1}}{l_{1t-1}}(\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}} - 1)A \qquad (1.5.1) \\ &\begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \end{aligned}$$

$$\frac{\partial l_{1t-1}}{\partial \Delta K_t} = \frac{1}{\Delta(t-1)}\beta V_{kk}(t) F_{1l} U_{cc} P_{t-1}(\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}} - 1)A \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \qquad (1.5.2)$$

$$\begin{aligned} \frac{\partial l_{2t-1}}{\partial \Delta K_t} &= \frac{1}{\Delta(t-1)}\beta V_{kk}(t) F_{1l} U_{cc} P_{t-1}\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}}(1 - \frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}})A \qquad (1.5.3) \\ &\begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \end{aligned}$$

$$\frac{\partial I_{t-1}}{\partial \Delta K_t} = \frac{\partial K_t}{\partial \Delta K_t} = -\frac{1}{\Delta(t-1)}\beta V_{kk}(t)(G_{zz} - F_{1l}^2 U_{cc})(\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}} - 1)^2 A < 0. \qquad (1.5.4)$$

$$\frac{\partial Z_{t-1}}{\partial \Delta K_t} = -\frac{\partial(l_{1t-1} + l_{2t-1})}{\partial \Delta K_t} = \frac{1}{\Delta(t-1)}\beta V_{kk}(t) F_{1l} U_{cc} P_{t-1}(\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}} - 1)^2 A > 0. \qquad (1.5.6)$$

$$\begin{aligned} \frac{\partial(y_{1t-1} + P_{t-1}y_{2t-1})}{\partial \Delta K_t} &= \frac{\partial F_1(t-1)}{\partial \Delta K_t} + P_{t-1}\frac{\partial F_2(t-1)}{\partial \Delta K_t} \\ &= -\frac{1}{\Delta(t-1)}\beta V_{kk}(t-1) F_{1l}^2 U_{cc} P_{t-1}(\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}} - 1)^2 A < 0 \qquad (1.5.7) \end{aligned}$$

$$\frac{\partial C_{t-1}}{\partial \Delta K_t} = \frac{\partial(y_{1t-1} + P_{t-1}y_{2t-1} - P_{t-1}I_{t-1})}{\partial \Delta K_t} = \frac{1}{\Delta(t-1)}\beta V_{kk}(t)G_{zz}\left(\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}} - 1\right)^2 A > 0 \tag{1.5.8}$$

$$\frac{\partial F_1(t-1)}{\partial \Delta K_t} = F_{1k}\frac{\partial k_{1t}}{\partial \Delta K_t} + F_{1l}\frac{\partial l_{1t}}{\partial \Delta K_t} = \frac{1}{\Delta(t-1)}V_{kk}(t)F_{1l}U_{cc}P_{t-1}\left(\frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}} - 1\right)\frac{F_1(t-1)}{l_{1t-1}}A \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \tag{1.5.9}$$

$$\frac{\partial F_2(t-1)}{\partial \Delta K_t} = F_{2k}\frac{\partial k_{2t}}{\partial \Delta K_t} + F_{2l}\frac{\partial l_{2t}}{\partial \Delta K_t} = \frac{1}{\Delta(t-1)}\beta V_{kk}(t)F_{1l}U_{cc}P_{t-1}\frac{k_{1t-1}}{l_{1t-1}}\left(1 - \frac{k_{1t-1}l_{2t-1}}{l_{1t-1}k_{2t-1}}\right)\frac{F_2(t-1)}{k_{2t-1}}A \begin{cases} > 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} < \frac{k_{2t-1}}{l_{2t-1}}; \\ < 0, & \text{if } \frac{k_{1t-1}}{l_{1t-1}} > \frac{k_{2t-1}}{l_{2t-1}}. \end{cases} \tag{1.5.10}$$

$$\left.\frac{\partial(\frac{k_{1t}}{l_{1t}})}{\partial P_t}\right|_{K_t} = [\beta V_{kk}(t+1)G_{zz} + \beta V_{kk}(t+1)F_{1l}^2 U_{cc} + P_t^2 U_{cc}G_{zz}] \cdot \frac{1}{\Delta(t)l_{1t}^2}P_t U_c^2 \frac{F_2(t)l_{1t}}{k_{2t}}\left(1 - \frac{k_{1t}l_{2t}}{l_{1t}k_{2t}}\right)F_{2ll} \begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}}; \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}. \end{cases} \tag{1.6.2}$$

$$\left.\frac{\partial(\frac{k_{2t}}{l_{2t}})}{\partial P_t}\right|_{K_t} = [\beta V_{kk}(t+1)G_{zz} + \beta V_{kk}(t+1)F_{1l}^2 U_{cc} + P_t^2 U_{cc}G_{zz}] \cdot \frac{1}{\Delta(t)l_{2t}^2}U_c^2 \frac{F_1(t)k_{2t}}{P_t l_{1t}}\left(1 - \frac{k_{1t}l_{2t}}{l_{1t}k_{2t}}\right)F_{1kk} \begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}}; \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}. \end{cases} \tag{1.6.3}$$

$$\left.\frac{\partial(F_1(k_{1t}, l_{1t}) - C_t)}{\partial H}\right|_{H=\bar{H}} = [F_{1k}(\beta V_{kk}(t+1) + P_t^2 U_{cc})\frac{\bar{k}_{1t}}{\bar{l}_{10}} + F_{1l}P_t^2 U_{cc} + F_{1l}\beta V_{kk}(t+1)\frac{\bar{k}_{1t}\bar{l}_{2t}}{\bar{l}_{1t}\bar{k}_{2t}}] \cdot \frac{G_{zz}}{\Delta(t)}\left(1 - \frac{\bar{k}_{1t}\bar{l}_{2t}}{\bar{l}_{1t}\bar{k}_{2t}}\right)A \begin{cases} > 0, & \text{if } \frac{k_{1t}}{l_{1t}} < \frac{k_{2t}}{l_{2t}}; \\ < 0, & \text{if } \frac{k_{1t}}{l_{1t}} > \frac{k_{2t}}{l_{2t}}. \end{cases} \tag{1.6.13}$$

$$\frac{\partial(F_2(k_{2t},l_{2t})-I_t)}{\partial H}\Big|_{H=\bar{H}}$$

$$= [F_{2k}\frac{\bar{k}_{1t}}{\bar{l}_{1t}}(\beta V_{kk}(t+1)+P_t^2U_{cc})+F_{2l}P_t^2U_{cc}+F_{2l}\beta V_{kk}(t+1)\frac{\bar{k}_{1t}\bar{l}_{2t}}{\bar{l}_{1t}\bar{k}_{2t}}]\cdot \tag{1.6.14}$$

$$\cdot\frac{G_{zz}}{\Delta(t)}(\frac{\bar{k}_{1t}\bar{l}_{2t}}{\bar{l}_{1t}\bar{k}_{2t}}-1)A\begin{cases}>0, & \text{if } \frac{\bar{k}_{1t}}{\bar{l}_{1t}}>\frac{\bar{k}_{2t}}{\bar{l}_{2t}};\\ <0, & \text{if } \frac{\bar{k}_{1t}}{\bar{l}_{1t}}<\frac{\bar{k}_{2t}}{\bar{l}_{2t}}.\end{cases}$$

Appendix II

Appendix to Chapter 3

II.1. Appendix to Section 3.4.

Proof of Lemma 3.4.1: Since $f(0) = g[S_2(0)] > 0$ and $f(\tau_u) = (1-c)\tau_u \leq \tau_u$, there exists $\tau_s \in [0, \tau_u]$ such that $\tau_s = f(\tau_s)$ by the intermediate value theorem. The concavity of f implies that the fixed point is unique.

From (3.3.20)

$$\lim_{p \to 0} \frac{dS_2}{dp} = +\infty \tag{II.1.1}$$

which implies that

$$\lim_{\tau \to \tau_u} \frac{df}{d\tau} = -\infty \tag{II.1.2}$$

Since $D_\tau f(0) > 0$, by the continuity of $D_\tau f(\tau)$ there exists a unique critical point, τ_m, for f. Since f is concave, $f(\tau_m)$ is a maximum.

Q.E.D.

Proof of Lemma 3.4.2: The concavity of f implies that $f(\tau) < f(\tau_s) + f'(\tau_s)(\tau - \tau_s)$. Since $f(\tau) > \tau$ for $\tau < \tau_s$, one can prove that $D_\tau f(\tau_s) < 1$, which implies that $D_\tau g(\tau_s) < c$.

τ_s satisfies $c\tau_s = g[S_2(\tau_s)]$. Total differentiation reveals:

$$\frac{d\tau_s}{dc} = \frac{\tau_s}{(D_\tau g(\tau_s) - c)} < 0 \tag{II.1.3}$$

The critical point of f satisfies $(1-c) = -D_\tau g(\tau_m)$. Total differentiation gives:

$$\frac{d\tau_m}{dc} = \left(\frac{d^2 g}{d\tau^2}\bigg|_{\tau_m}\right)^{-1} < 0 \tag{II.1.4}$$

Since $\tau_m' = (1-c)\tau_m + g(\tau_m)$, the envelope theorem implies:

$$\frac{d\tau_m'}{dc} = -\tau_m < 0 \tag{II.1.5}$$

Q.E.D.

Proof of Lemma 3.4.3: When $c = 1$, $f(\tau) = g(\tau)$ and $\tau_m = \dot{\tau}$. Since $g(S_2) < g(b)$, (A3.4) implies that at $c = 1$, $\tau_s < \tau_m$. When $c = 0$, $\tau_s = \tau_u$. $D_\tau g(\tau_m(0)) = 1$ implies that $\tau_m(0)$ is in the interior of $[0,\ \tau_u]$. Hence $\tau_s > \tau_m$ at $c = 0$.

Since $\tau_s < \tau_m$ at $c = 1$ and $\tau_s > \tau_m$ at $c = 0$, the continuity of $\tau_m(c)$ and $\tau_s(c)$ implies that there exists at least one $c^1 \in (0,\ 1)$ such that $\tau_m(c) = \tau_s(c) = \tau'_m(c)$. Such c is unique because

$$\frac{d(\tau_s - \tau_m)}{dc}\Big|_{\tau_s = \tau_m} = -\tau_m - \frac{1}{D_\tau^2 g(\tau_m)} < 0. \qquad (II.1.6)$$

Otherwise, $\tau_s(c) - \tau_m(c)$ will be upward-sloping for at least one of such c.

Therefore, by the continuity of τ_m and τ_s in c, $\tau_s > \tau_m$ for $c \in (0,\ c^1)$ and $\tau_s < \tau_m$ for $c \in (c^1,\ 1]$.

Notice that $\tau_s > \tau_m$ implies that $f(\tau_m) > \tau_m$, ie, $\tau'_m > \tau_m$. Similarly, $\tau_s < \tau_m$ implies that $\tau'_m < \tau_m$.

Q.E.D.

Proof of Lemma 3.4.4: $\tau_s(0) \neq \tau_m(0)$. Thus, $\tau'_m(0) > \tau_s(0) = \tau_u$ by the definition of τ'_m. At c^1, $\tau'_m = \tau_m < \tau_u$. (II.1.5) implies that there exists a unique $c^0 \in (0\ \ c^1)$ such that $\tau'_m = \tau_u$.

Q.E.D.

Proof of Lemma 3.4.6: Comparative statics gives:

$$\frac{d(f(\tau'_m) - \tau_m)}{dc} = -f'(\tau'_m)\tau_m - \tau'_m - \frac{1}{D_\tau^2 g(\tau_m)} \qquad (II.1.7)$$

Since at $c = c^1$, $\tau_m = \tau'_m = \tau_s$ and $-\tau_m D_\tau^2 g(\tau_m) > 1$, then $\frac{d(f(\tau'_m) - \tau_m)}{dc}\big|_{c=c^1} < 0$. Hence there exists $c_f \in [c^0,\ c^1)$ such that $f(\tau'_m) > \tau_m\ \forall c \in (c_f,\ c^1)$.

Q.E.D.

II.2. Definitions and Theorems

This appendix presents definitions and theorems in the theory of one-dimensional dynamical system that are used in this chapter. Most of them are drawn from Collet and Eckmann (1980) (CE) and Grandmont (1986). The sources are identified for each definitions and theorems. Their notations are changed to conform with the notations in this chapter.

Definition II.1 (Grandmont). *A map $f : [0,\ \tau'_m] \to [0,\ \tau'_m]$ is* <u>*unimodal*</u> *if*

(1) f is continuous;

(2) there exists τ_m in $(0,\ \tau'_m)$ such that f is increasing on $[0,\ \tau_m]$ and decreasing on $[\tau_m,\ \tau'_m]$;

(3) $f(\tau_m) = \tau'_m$.

Definition II.2 (Grandmont). *A unimodal mapping $f(\tau)$ is* <u>*C^1-unimodal*</u> *if f is once continuously differentiable and $f'(\tau) \neq 0$ when $\tau \neq \tau_m$.*

Definition II.3 (CE). *A unimodal mapping $f(\tau)$ is* <u>*S-unimodal*</u> *if*

(1) f is C^3;

(2) $S(f) < 0$ whenever $f'(\tau) \neq 0$, where $S(f)$ denotes the Schwarzian derivative;

(3) f maps $J(f) = [f(\tau'_m), \tau'_m]$ onto itself;

(4) $f''(\tau_m) < 0$.

Definition II.4 (CE). *Let $f : [0,\ \tau'_m] \to [0,\ \tau'_m]$ be a unimodal mapping. The* <u>*itinerary*</u> *associated with $\tau \in [0,\ \tau'_m]$, denoted by $\underline{I}(\tau)$, is a finite or infinite sequence of the symbols L, C, R that satisfies:*

(1) $\underline{I}(\tau)$ is either an infinite sequence of L's and R's, or a finite (or empty) sequence of L's and R's, followed by C. The j^{th} element of $I(\tau)$ will be denoted $I_j(\tau)$, $j = 0, 1, \ldots$;

(2) If $f^j(\tau) \neq \tau_m$ for all $j \geq 0$, then $I_j(\tau) = L$ if $f^j(\tau) < \tau_m$ and $I_j(\tau) = R$ if $f^j(\tau) > \tau_m$;

(3) If $f^k(\tau) = \tau_m$ for some k, then letting j denote the smallest such k, $I_j(\tau) = C$ and $I_l(\tau) = L$ if $0 \leq l < j$ and $f^l(\tau) < \tau_m$ and $I_l(\tau) = R$ if $0 \leq l < j$ and $f^l(\tau) > \tau_m$.

Definition II.5 (CE). *A sequence $\underline{I}$ of symbols L, C, R is called admissible if either $\underline{I}$ is an infinite sequence of L's and R's or if $\underline{I}$ is a finite (or empty) sequence of L's and R's, followed by C.*

By definition II.4., every itinerary is an admissible sequence. Given two sequences $\underline{I}$ and $\underline{J}$, $\underline{IJ}$ denotes the concatenation of $\underline{I}$ and $\underline{J}$. $\underline{I}^n = \underline{I}...\underline{I}$ (n times) and $\underline{I}^\infty = \underline{II}...$ infinitely. $|\underline{I}|$ denotes the cardina[illegible] of $\underline{I}$. The word *Kneading sequence* refers to $\underline{I}(\tau'_m)$.

Definition II.6 (CE). *The extended itinerary is defined as follows:*

(1) $\underline{I}_E(\tau) = \underline{I}(\tau)$ *if* $\underline{I}(\tau)$ *is infinite;*

(2) $\underline{I}_E(\tau) = \underline{I}(\tau)\underline{I}(\tau'_m)$ *if* $\underline{I}(\tau)$ *is finite and* $\underline{I}(\tau'_m)$ *is infinite;*

(3) $\underline{I}_E(\tau) = \underline{I}(\tau)(\underline{I}(\tau'_m))^\infty$ *if both* $\underline{I}(\tau)$ *and* $\underline{I}(\tau'_m)$ *are finite.*

Definition II.7 (CE). *Given two admissible sequences* $\underline{A}$ *and* $\underline{B}$, $\underline{A} < \underline{B}$ *if either*

(1) there are an even number of R's in $A_0A_1...A_{i-1} = B_0B_1...B_{i-1}$ *and* $A_i < B_i$*;*

or

(2) there are an odd number of R's in $A_0A_1...A_{i-1}$ *and* $A_i > B_i$.

It can be shown that the ordering defined above is complete.

Definition II.8 (CE). *A finite sequence* $\underline{B}$ *is said to be even if it has an even number of R's. It is said to be odd otherwise.*

Given $\underline{I} = I_0I_1...$, the shift operation ρ is defined as $\rho\underline{I} = I_1I_2I_3...$. If $\underline{I} = C$, ρ is undefined.

Definition II.9. *Let* $\{\tau_1, ..., \tau_k\}$ *be a periodic orbit of a unimodal map* $f(\tau)$ *that maps* $[0, \tau'_m]$ *into itself. A point* $\tau \in [0, \tau'_m]$ *is said to be attracted to the periodic orbit if* $\lim_{m\to\infty} f^{mk}(\tau) = \tau_i$ $(i = 1$, *or* 2, ..., *or* $k)$.

Definition II.10 (CE). *An admissible sequence* $\underline{A}$ *is dominated by* $\underline{I}(\tau'_m)$*, denoted by* $\underline{A} << \underline{I}(\tau'_m)$*, if for all* $k \geq 0$

(1) $\rho^k \underline{A} < \underline{I}(\tau_m')$ *if* $\underline{I}(\tau_m')$ *is infinite;*

(2) $\rho^k \underline{A} < (\underline{D}L)^\infty$ *if* $\underline{I}(\tau_m') = \underline{D}C$ *and* $\underline{D}$ *is even;*

(3) $\rho^k \underline{A} < (\underline{D}R)^\infty$ *if* $\underline{I}(\tau_m') = \underline{D}C$ *and* $\underline{D}$ *is odd.*

Definition II.11 (CE). *Given a unimodal mapping* $f : [0, \tau_m'] \to [0, \tau_m']$, *let* W *be an open subset of* $(0, \tau_m) \cup (\tau_m, \tau_m')$. $f|_W$ *is said to have a* <u>*sink*</u> *if there is an open interval* $K \subset W$ *such that* $f^n(K) \subset K$ *for some* $n \geq 1$ *and* $f^j(K) \subset W$ *for* $j = 1, \ldots, n-1$.

Definition II.12 (CE). *A one-parameter family of maps* f_c: $c \to f_c$ *is a map from* $[\underline{c}, \bar{c}]$ *to the* C^1*-unimodal maps such that* $\sup_{\tau \in [0, \tau_m']} |f_c(\tau) - f_{\dot{c}}(\tau)| + |f_c'(\tau) - f_{\dot{c}}'(\tau)| \to 0$ *when* $c \to \dot{c}$.

Let $\underline{K}(f_c)$ denote the itinerary $\underline{I}_{f_c}(\tau_m')$.

Definition II.13 (Grandmont). *A family of unimodals* f_c *is full if* $K(f_{\underline{c}}) = RLL\ldots$ *and* $K(f_{\bar{c}}) = R^\infty$.

The following results are drawn from CE and Grandmont (1986). Each result is identified by the number in the original sources.

CE Lemma II.3.1. *If* f *is unimodal then* $\underline{I}_E(\tau)$ *is eventually periodic if and only if* $f^j(\tau)$ *converges towards a periodic orbit of* f *as* $j \to \infty$.

CE Lemma II.3.2. *If* f *is unimodal and* $\underline{I}_E(\tau)$ *has eventual period* p, $\underline{I}_E(\tau) = \underline{A}\underline{B}^\infty$ *with* $|\underline{B}| = p$ *then the periodic orbit of* f *towards which* $f^j(\tau)$ *converges has period* p *if* $\underline{B}$ *is even and period* p *or* $2p$ *if* $\underline{B}$ *is odd.*

CE Theorem II.3.8. *Let* f *be unimodal, and assume* $\underline{A}$ *is an admissible sequence satisfying* $\underline{I}(0) \leq \underline{A} << \underline{I}(\tau_m')$. *Then there is a* $\tau \in [0, \tau_m']$ *such that* $\underline{I}(\tau) = \underline{A}$.

CE Lemma II.5.1. *If* f *is* C^1*-unimodal and if* W *is an open subset of* $(0, \tau_m) \cup (\tau_m, \tau_m')$ *such that* $f|_W$ *has a sink, then* $\bar{W}$ *contains a weakly stable periodic orbit of* f.

CE Proposition II.5.7. *Assume f is S-unimodal and has a weakly stable periodic orbit. Let E be the set of τ in $[0, \tau'_m]$ such that $f^n(\tau)$ does not converge to the stable periodic orbit of f. Then $\mu(E) = 0$.*

CE Theorem II.7.1. *If f is S-unimodal and has a weakly stable periodic orbit of period k, then f has no sensitivity.*

Grandmont Proposition 3. *Assume that f is S-unimodal. f has a unique weakly stable cycle P if and only if $\underline{I}_E(\tau'_m)$ is periodic. If the period of $\underline{I}_E(\tau'_m)$ is k, the period of P is k or $2k$.*

II.3. Appendix to Sections 3.6 and 3.7

Proof of Proposition 3.6.1: $c_f = c^0$ implies that $f(\tau'_m) > \tau_m$ for $c \in (c^0, c^1)$. Hence the extended itinerary of τ_0, $\underline{I}_E(\tau_0) = \underline{A}R^\infty$, is eventually periodic. Since the sequence of $\{R\}$ contains an odd number of R, the conclusion follows by applying CE II.3.1 and II.3.2.

Q.E.D.

Proof of Proposition 3.6.2: Define two identical open sets, $K = W = (f(\tau'_m), \tau'_m)$. Since for $\tau_t \in (f(\tau'_m), \tau'_m)$, $f(\tau_t) \in (f(\tau'_m), \tau'_m)$,

$$f^n(K) \subset (f(\tau'_m), \tau'_m) = K = W$$

$\forall n \geq 1$. Thus $f_{|W}$ has a sink. Thus, $\bar{W} = [f(\tau'_m), \tau'_m]$ contains a weakly stable period orbit by CE II.5.1.

Q.E.D.

Proof of Proposition 3.6.4: Proposition 3.6.1 states that all possible periodic orbits are of period one or two. Hence we can concentrate on the graph of $\varphi(\tau) \equiv f^2(\tau)$.

Define τ_k as the smaller of τ that satisfies $f(\tau) = \tau_m$. ($\tau_k < \tau_m$). Then $\varphi(\tau)$ is increasing in the interval $[0, \tau_k)$ and decreasing in (τ_k, τ_m) with τ_k being a local maximum of $\varphi(\tau)$. Similarly $\varphi(\tau)$ is increasing in the interval $[\tau_m, \tau'_m]$ with τ_m being the local minimum of $\varphi(\tau)$. Figure II.3.1 displays the shape of $\varphi(\tau)$. $\varphi(\tau)$ has at least one fixed point, namely τ_s.

Suppose f has more than two periodic orbits. Then $\varphi(\tau)$ must have at least three fixed points in the interval $(\tau_m, \tau_s]$. Let τ^1, τ^2 and τ^3 with $\tau_m < \tau^1 < \tau^2 < \tau^3 \leq \tau_s$ be three consecutive fixed points of φ. As proven in CE p98, $S(f) < 0$ implies that $D\varphi(\tau^2) > 1$. Since $D\varphi(\tau) > 0$ for $\tau \in [\tau_m, \tau_s]$, it must be true that $0 < D\varphi(\tau^1) < 1$ and that $0 < D\varphi(\tau^3) \leq 1$, which implies that $\varphi(\tau)$ has more than one weakly stable periodic orbits; a contradiction to Proposition 3.6.3.

Proof of (a): Let τ^1 denote the smallest fixed point of φ. $\tau^1 > \tau_m$ since $\varphi(\tau_m) = f(\tau'_m) > \tau_m$. τ^1 belongs to a weakly stable periodic orbit because $0 < D\varphi(\tau^1) \leq 1$.

$-D_\tau g(\tau_s) \leq 2 - c$ implies that $-D_\tau f(\tau_s) \leq 1$, ie, τ_s is weakly stable. Then $\tau_s = \tau^1$ by Proposition 3.6.3. It is impossible that $\varphi(\tau)$ has other fixed point by the definition of τ^1.

Proof of (b): $-D_\tau g(\tau_s) > 2 - c$ implies that $-D_\tau f(\tau_s) > 1$, ie, τ_s is unstable. Hence τ_1 must belong to a weakly stable two-period cycle.

Q.E.D.

Proof of Proposition 3.6.7:

$$\begin{aligned} f(\tau'_m) - \tau_m &= (1-c)\tau'_m + g(\tau'_m) - \tau_m \\ &= -c\tau'_m + g(\tau'_m) + \tau'_m - \tau_m \qquad (II.3.1) \\ &= [g(\tau'_m) - c\tau'_m] + [g(\tau_m) - c\tau_m] \end{aligned}$$

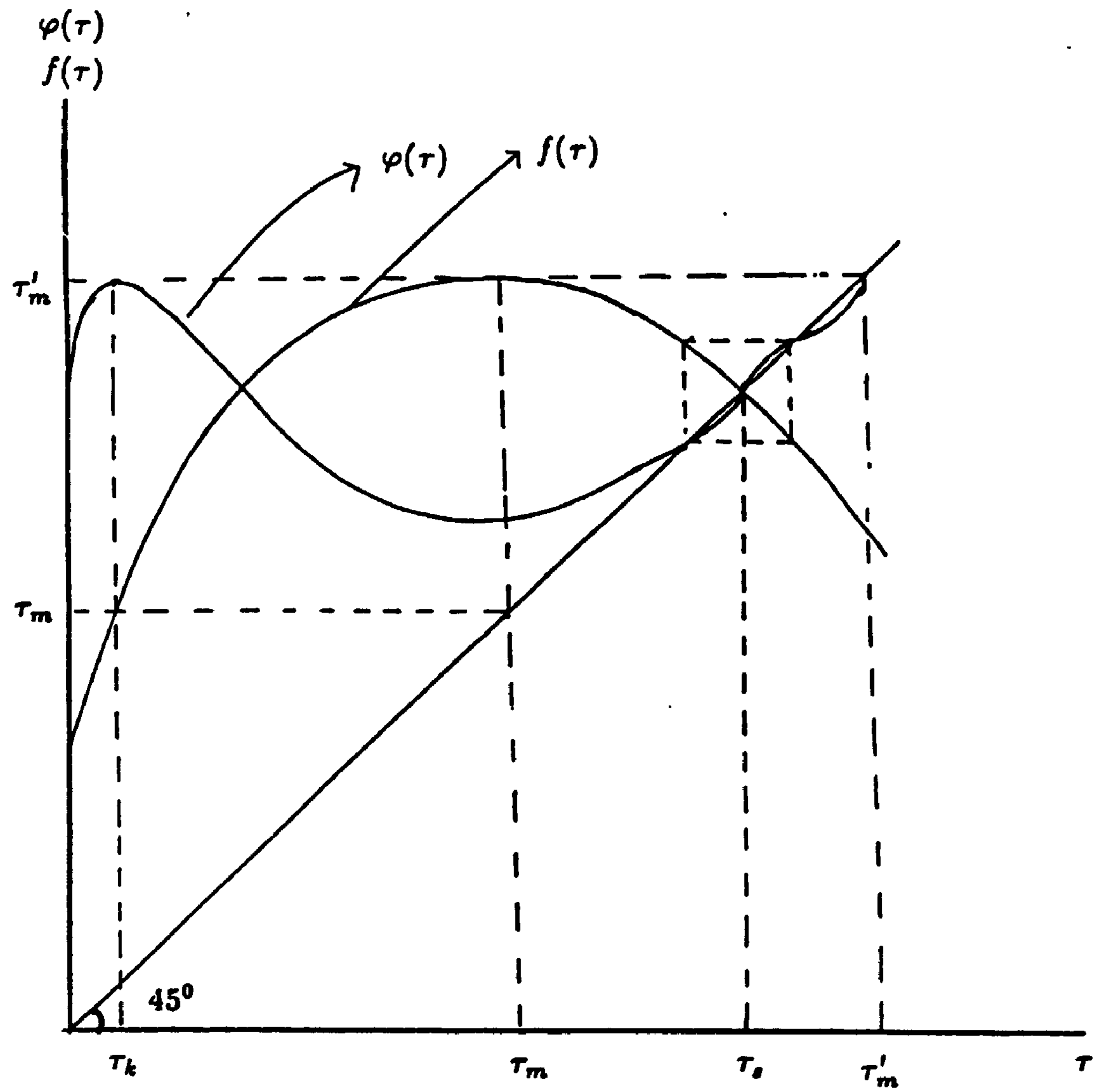
The result follows from (II.3.1) by utilizing the fact that $g(\tau_m) > c\tau_m$ for $c \in (c^0, c^1)$.

Q.E.D.

Proof of Corollary 3.6.8: For $c \in (c^0, c^1)$, $\tau_m \in (\tau_m(c^1), \tau_m(c^0))$.

Since $(1-c) + g'(\tau_m) = 0$,

$$c = 1 + g'(\tau_m) \qquad (II.3.2)$$



$\varphi(\tau)$
$f(\tau)$
$\varphi(\tau)$
$f(\tau)$
τ'_m
τ_m
45^0
τ_k
τ_m
τ_s
τ'_m
τ

FIGURE II.3.1.

Therefore,

$$\begin{aligned} \tau_m' &= -g'(\tau_m)\tau_m + g(\tau_m) \\ &= g(\tau_m)[1 + \eta_{g\tau}(\tau_m)] \end{aligned} \qquad (II.3.3)$$

(3.6.2) is obtained by substituting (II.3.2) for c and (II.3.3) for τ_m' into (3.6.1).

Q.E.D.

Proof of Proposition 3.7.4: $f(\tau_m) \equiv \tau_m' > \tau_m$ and $f(\tau_m') < \tau_m$ since $c \in (c^0, c_f)$. Take Taylor expansion of $f[f(\tau_m')]$ around τ_m and notice that $D_\tau^2 f(\tau) = D_\tau^2 g(\tau)$:

$$f^2(\tau_m') - \tau_m = f(\tau_m) - \tau_m + \frac{D_\tau^2 g(v)}{2}[f(\tau_m') - \tau_m]^2 < 0 \qquad (II.3.4)$$

for $v \in [f(\tau_m'), \tau_m]$ under the condition given in the proposition.

Therefore, $f^2(\tau_m') < \tau_m$, ie $f(\tau_m') < \tau_k$, given this particular c.

Q.E.D.

Appendix III

Appendix to Chapter 4

Proof of Lemma 4.5.1.: $\tilde{\alpha} < 1$ because $y^{\frac{1}{\gamma}} < 1$. Define a function

$$f(\gamma) \equiv x^{\frac{1}{\gamma}} + y^{\frac{1}{\gamma}} \tag{III.1}$$

Then

$$f'(\gamma) = -\frac{1}{\gamma^2} x^{\frac{1}{\gamma}} \ln x - \frac{1}{\gamma^2} y^{\frac{1}{\gamma}} \ln y < 0 \tag{III.2}$$

because the function $z^{\frac{1}{\gamma}} \ln z$ is increasing in z and $x > y$.

Since $f(1) = x + y > 2$, $f(\gamma) > 2$ for $\gamma \in (0,\ 1)$, which implies that $\tilde{\alpha} > \frac{1}{2}$.

Q.E.D.

Proof of Proposition 4.5.2.: Since

$$F(W) - (1-\alpha)^{\gamma} \bar{u} = (1-\alpha)^{\gamma} \beta W [x - (\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha} y^{\frac{1}{\gamma}})^{\gamma}] \begin{cases} > 0, & \text{if } \alpha < \tilde{\alpha}; \\ = 0, & \text{if } \alpha = \tilde{\alpha}; \\ < 0, & \text{if } \alpha > \tilde{\alpha}, \end{cases} \tag{III.3}$$

the result follows from (4.4.10) and (III.3).

Q.E.D.

Proof of Proposition 4.5.3.: Using (4.4.9), one can obtain:

$$\begin{aligned} &\frac{(1-\alpha)\hat{\pi}^{\frac{1}{\gamma}}}{(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{1-\gamma}} - \frac{\beta(1-\alpha)(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}}}{(W^{\frac{1}{\gamma}} - \alpha(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}})^{1-\gamma}} - (1-\alpha)^{\gamma}\bar{u} \\ &= \frac{(W^{\frac{1}{\gamma}} - (\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}})}{(W^{\frac{1}{\gamma}} - \alpha(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}})^{1-\gamma}} - \frac{(W^{\frac{1}{\gamma}} - \hat{\pi}^{\frac{1}{\gamma}})}{(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{1-\gamma}} \\ &> 0 \end{aligned} \tag{III.4}$$

The sign of (III.4) is determined by that $(W^{\frac{1}{\gamma}} - z^{\frac{1}{\gamma}})(W^{\frac{1}{\gamma}} - \alpha z^{\frac{1}{\gamma}})^{\gamma-1}$ is decreasing in z and that $\hat{\pi} > \bar{u} + \beta\hat{\pi}$.

$$\frac{d\hat{\pi}}{d\alpha} = -\frac{\gamma}{\alpha} \cdot \frac{\frac{\hat{\pi}^{\frac{1}{\gamma}}}{(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{1-\gamma}} - \frac{\beta(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}}}{(W^{\frac{1}{\gamma}} - \alpha(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}})^{1-\gamma}} - (1-\alpha)^{\gamma-1}\bar{u}}{\hat{\pi}^{\frac{1}{\gamma}-1}(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{\gamma-1} - \beta^2(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}-1}(W^{\frac{1}{\gamma}} - \alpha(\bar{u} + \beta\hat{\pi})^{\frac{1}{\gamma}})^{\gamma-1}} < 0 \tag{III.5}$$

The sign of the numerator of (III.5) is determined by (III.4) while the sign of denominator is determined by that $\hat{\pi} > \bar{u} + \beta\hat{\pi}$.

Q.E.D.

Proof of Proposition 4.5.4.: The welfare possibility frontier (4.3.24) implies that $\hat{\pi} < \alpha^{-\gamma}W$. Combining it with $\bar{u} < (1-\beta)\alpha^{1-\gamma}W$ gives

$$W^{\frac{1}{\gamma}} > \beta W^{\frac{1}{\gamma}} + \alpha^{\gamma-1}W^{\frac{1-\gamma}{\gamma}}\bar{u} > \beta W^{\frac{1}{\gamma}} + \hat{\pi}^{\frac{1-\gamma}{\gamma}}\bar{u} \tag{III.6}$$

Using (4.3.24) and (III.5) one obtains:

$$\begin{aligned}\frac{d\hat{\pi}^*}{d\alpha} =& \frac{\gamma(1-\alpha)^{-\gamma}(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{\gamma-1}[W^{\frac{1}{\gamma}} - \alpha(\bar{u}+\beta\hat{\pi})^{\frac{1}{\gamma}}]^{\gamma-1}}{\hat{\pi}^{\frac{1}{\gamma}-1}(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{\gamma-1} - \beta^2(\bar{u}+\beta\hat{\pi})^{\frac{1}{\gamma}-1}(W^{\frac{1}{\gamma}} - \alpha(\bar{u}+\beta\hat{\pi})^{\frac{1}{\gamma}})^{\gamma-1}} \cdot \\ & \cdot [\pi^{\frac{1-\gamma}{\gamma}}W^{\frac{1}{\gamma}} - (\bar{u}+\beta\hat{\pi})^{\frac{1-\gamma}{\gamma}}(\beta W^{\frac{1}{\gamma}} + \hat{\pi}^{\frac{1-\gamma}{\gamma}}\bar{u})] \\ & > 0\end{aligned} \tag{III.7}$$

Q.E.D.

Proof of Proposition 4.5.5.: By (III.5), one has

$$\frac{d(\alpha\hat{\pi})}{d\alpha} = \hat{\pi} + \alpha\frac{d\hat{\pi}}{d\alpha} > 0. \tag{III.8}$$

$$\begin{aligned}\frac{d((1-\alpha)\hat{\pi}^*)}{d\alpha} =& -(1-\gamma)(1-\alpha)^{-\gamma}(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{\gamma} \\ & -(1-\alpha)^{1-\gamma}\gamma(W^{\frac{1}{\gamma}} - \alpha\hat{\pi}^{\frac{1}{\gamma}})^{\gamma-1}\hat{\pi}^{\frac{1-\gamma}{\gamma}}(\hat{\pi} + \frac{\alpha}{\gamma}\frac{d\hat{\pi}}{d\alpha}) \\ & < 0\end{aligned} \tag{III.9}$$

Q.E.D.

Proof of Lemma 4.5.6.: $\Delta(\alpha) = 0$ requires that

$$\hat{\pi} = \frac{W}{\beta}[\frac{(1-\alpha)^{\frac{\gamma}{1-\gamma}}}{(\alpha\beta)^{\frac{1}{1-\gamma}} + \alpha(1-\alpha)^{\frac{\gamma}{1-\gamma}}}]^{\gamma} - \frac{1}{\beta}\bar{u} \tag{III.10}$$

Substitute (III.10) for π in (4.4.9), one has:

$$\begin{aligned}&(W^{\frac{1}{\gamma}} - \frac{1}{\beta^{\frac{1}{\gamma}}}[(\frac{(1-\alpha)^{\frac{\gamma}{1-\gamma}}}{\alpha^{\frac{\gamma}{1-\gamma}}\beta^{\frac{1}{1-\gamma}} + (1-\alpha)^{\frac{\gamma}{1-\gamma}}})^{\gamma}W - \alpha^{\gamma}\bar{u}]^{\frac{1}{\gamma}})^{\gamma} \\ & - \beta W\frac{\alpha^{\frac{\gamma^2}{1-\gamma}}\beta^{\frac{\gamma}{1-\gamma}}}{(\alpha^{\frac{\gamma}{1-\gamma}}\beta^{\frac{1}{1-\gamma}} + (1-\alpha)^{\frac{\gamma}{1-\gamma}})^{\gamma}} = (1-\alpha)^{\gamma}\bar{u}\end{aligned} \tag{III.11}$$

If $\alpha = 0$, the left-hand-side of (III.11) is less than the right-hand-side. If $\alpha = 1$, the reverse is true. Therefore, there exists $\hat{\alpha} \in (0,\ 1)$ such that (III.11) holds.

Q.E.D.

Proof of Lemma 4.5.7.: Using (III.5), one can verify that

$$\bar{u} + \beta\hat{\pi} + \frac{\alpha\beta}{\gamma}\frac{d\hat{\pi}}{d\alpha} > 0 \qquad (III.12)$$

Since $\frac{d\hat{\pi}}{d\alpha} < 0$, (III.12) implies that

$$\bar{u} + \beta\hat{\pi} + \frac{\alpha\beta(1-\gamma)}{\gamma}\frac{d\hat{\pi}}{d\alpha} > 0 \qquad (III.13)$$

Therefore,

$$\begin{aligned}\frac{d\Delta}{d\alpha} =& \gamma(1-\alpha)^{\gamma-1}[W^{\frac{1}{\gamma}} - \alpha(\bar{u}+\beta\hat{\pi})^{\frac{1}{\gamma}}]^{1-\gamma} \\ &+ (1-\gamma)(1-\alpha)^{\gamma}[W^{\frac{1}{\gamma}} - \alpha(\bar{u}+\beta\hat{\pi})^{\frac{1}{\gamma}}]^{-\gamma}(\bar{u}+\beta\hat{\pi})^{\frac{1-\gamma}{\gamma}}(\bar{u}+\beta\hat{\pi}+\frac{\alpha\beta}{\gamma}\frac{d\hat{\pi}}{d\alpha}) \\ &+ \beta(\bar{u}+\beta\hat{\pi})^{\frac{1-\gamma}{\gamma}}(\bar{u}+\beta\hat{\pi}+\frac{\alpha\beta(1-\gamma)}{\gamma}\frac{d\hat{\pi}}{d\alpha}) \\ &> 0\end{aligned} \qquad (III.14)$$

Since Δ is monotonic in α, $\hat{\alpha}$ is unique.

It can be verified that $\Delta(\frac{1}{2}) < 0$ by using that $\hat{\pi} > W$ when $\alpha = \frac{1}{2}$ and that $\hat{\pi} \geq \bar{u} + \beta\hat{\pi}$. (III.14) then implies that $\frac{1}{2} < \hat{\alpha}$.

Q.E.D.

Proof of Proposition 4.5.8.:

$$\frac{d\hat{\pi}}{d\bar{u}} = \frac{\Delta}{\alpha\hat{\pi}^{\frac{1-\gamma}{\gamma}}[\frac{W^{\frac{1}{\gamma}}-\alpha(\bar{u}+\beta\hat{\pi})^{\frac{1}{\gamma}}}{W^{\frac{1}{\gamma}}-\alpha\hat{\pi}^{\frac{1}{\gamma}}}]^{1-\gamma} - \alpha\beta^2(\bar{u}+\beta\hat{\pi})^{\frac{1-\gamma}{\gamma}}} \begin{cases} > 0, & \text{if } \alpha > \hat{\alpha}; \\ = 0, & \text{if } \alpha = \hat{\alpha}; \\ < 0, & \text{if } \alpha < \hat{\alpha}. \end{cases} \qquad (III.15)$$

The signs of $\frac{d\hat{\pi}^*}{d\bar{u}}$ are opposite to that of $\frac{d\hat{\pi}}{d\bar{u}}$ since $\frac{d\pi^*}{d\pi} < 0$.

Q.E.D.

Proof of Proposition 4.5.9.: The case where $\gamma = 1$ is obvious. For the case $\gamma \in (0,\ 1)$, substitute (4.5.8) for α in (4.5.9):

$$\Delta(\hat{\alpha}) = \frac{W^{\frac{1-\gamma}{\gamma}}}{x^{\frac{1}{\gamma}} - y^{\frac{1}{\gamma}}}(\beta y^{\frac{1}{\gamma}-1}(x^{\frac{1}{\gamma}} - 1) - x^{\frac{1}{\gamma}-1}(1 - y^{\frac{1}{\gamma}})) < 0. \qquad (III.16)$$

The sign of $\beta y^{\frac{1}{\gamma}-1}(x^{\frac{1}{\gamma}}-1)-x^{\frac{1}{\gamma}-1}(1-y^{\frac{1}{\gamma}})$ is determined by the second-order Taylor expansions of $x^{\frac{1}{\gamma}}$ and $y^{\frac{1}{\gamma}}$ around 1.

Q.E.D.

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